AN INTRODUCTORY TREATISE

ON

DYNAMICAL ASTRONOMY

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BY

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PREFACE

This book is intended to provide an introduction to those parts of Astronomy which require dynamical treatment. To cover the whole of this wide subject, even in a preliminary way, within the limits of a single volume of moderate size would be manifestly impossible. Thus the treatment of bodies of definite shape and of deformable bodies is entirely excluded, and hence no reference will be found to problems of geodesy or the many aspects of tidal theory. Already the study of stellar motions is bringing the methods of statistical mechanics into use for astronomical purposes, but this development is both too recent and too distinct in its subject-matter to find a place here.

Nevertheless the book covers a wider range of subject than has been usual in works of the kind. Thereby two advantages may be gained. For the reader is spared the repetition of very much the same introductory matter which would be necessary if the different branches of the subject were taken up separately. But in the second place, and this is more important, he will see these branches in due relation to one another and will realize better that he is dealing not with several distinct problems but with different parts of what is essentially a single problem. In an introductory work it therefore seemed desirable to make the scope as wide as was compatible with a reasonable unity of method, the more so on account of the almost complete absence of similar works in the English language.

The first six chapters are devoted to preliminary matters, chiefly connected with the undisturbed motion of two bodies. These are followed by five chapters VII to XI dealing with the determination of orbits. This section is intended to familiarize the reader with the properties of undisturbed motion by explaining in general terms the most important and interesting applications. It is in no sense complete and is not intended to replace those works which are entirely devoted to this subject. Otherwise it would have been necessary to describe in detail such admirably effective methods as Professor Leuschner's and to include fully worked numerical examples. Here, as elsewhere, the aim has been to give such an account of principles as will be

instructive to the reader whose studies in this branch go no further, and at the same time one which will help the student to understand more easily the technical details to be met with in more special treatises. Though the actual details of practical computation are entirely excluded, the fact that all such methods end in numerical application has by no means been overlooked. A distinct effort has been made to leave no formulae in a shape unsuitable for translation into numbers. The student who feels the need will have no difficulty in finding forms of computation in other works. At the same time the reader who will take the trouble to work out such forms for himself will be rewarded with a much truer mastery of the subject, though he should not disdain what is to be learnt from the tradition of practical computers.

An outline of the Planetary Theory is given in the seven chapters XII to XVIII. The first of these deals exclusively with the abstract dynamical principles which are subsequently employed. It is hoped that this synopsis will prove useful in avoiding the necessity for frequent reference to works on theoretical mechanics. The reader to whom the methods are unfamiliar and who wishes to become more fully acquainted with them may be referred to Professor Whittaker's *Analytical Dynamics*, where he will also find an introduction to those more purely theoretical aspects of the Problem of Three Bodies which find no place here. To those who are familiar with these principles in their abstract form only the concrete applications in the following chapters may prove interesting. A chapter on special perturbations is included. Here, as in the determination of orbits, the need for numerical examples may be felt. To have inserted them would have interfered too much with the general plan of the book, and they will be found in the more special treatises. But it was felt that the subject could not be omitted altogether, and a concise and fairly complete account of the theory has therefore been given. It may seem curious that with the development of analytical resources the need for these mechanical methods becomes greater rather than less, but so it is.

Chapter XIX on the restricted problem of three bodies is intended as an introduction to the Lunar Theory contained in Chapters XX and XXI. The division of these two chapters is partly arbitrary, for the sake of preserving a fair uniformity of length, but it coincides roughly with the distinction between Hill's researches and the subsequent development by Professor Brown. In the second a low order of approximation is worked out, and it is hoped that this will serve to some extent the double purpose of making the

whole method clearer and of pointing out the nature of the principal terms, which are apt to be entirely hidden by the complicated machinery of the systematic development.

The rotation of the Earth and Moon is discussed in Chapters XXII and XXIII. The treatment of precession and nutation is meant to be simple and practical, and the opportunity is taken to add an account of the astronomical methods of reckoning time in actual use. In the final chapter of the book the theory of the ordinary methods of numerical calculation is explained. This is necessary for the proper understanding of Chapter XVIII, but it also bears on various points which occur elsewhere. Numerical applications find no place in this work. But let the mathematical reader be under no misapprehension. The ultimate aim of all theory in Astronomy is seldom attained without comparison with the results of observation, and the medium of comparison is numerical. Hence few parts of the theory can be regarded as complete till they are reduced to a numerical form. This is a process which often demands immense labour and in itself a quite special kind of skill. It is just as essential as the manipulation of analytical forms.

Originality in the wider sense is not to be expected and indeed would defeat the object of the book, which aims at making it easier for the student to read with profit the larger and more technical treatises and to proceed to the original memoirs. A certain freshness in the manner of treatment is possible and, it is hoped, will not be found altogether wanting. Few direct references have been given as a guide to further reading, and this may be regretted. But the opinion may be expressed that for the reader who is qualified to profit by a work like the present, and who wishes to go further, the time has come when he should acquire, if he has not done so already, the faculty of consulting the library for what he wants without an apparatus of special directions. Sign-posts have their uses, and the experienced traveller is the last to despise them, but they are not conducive to a spirit of original adventure.

Since the main object in view has been to cover a wide extent of ground in a tolerably adequate way rather than to delay over critical details, the absence of mathematical rigour may sometimes be noticed. Very little attention is given to such questions as the convergence of series. It is not to be inferred that these points are unimportant or that the modern astronomer can afford to disregard them. But apart from a few simple cases where the

reader will either be able to supply what is necessary for himself, or would not benefit even if a critical discussion were added, such questions are extremely difficult and have not always found a solution as yet. It is precisely one of the aims of this book to increase the number of those who can appreciate this side of the subject and will contribute to its elucidation.

The reader who wishes to proceed further in any parts of the subject to which he is introduced in this book will soon find that the number of systematic treatises available in all languages is by no means large. He must turn at an early stage to the study of original memoirs. It is not difficult to find assistance in such sources as the articles in the *Encyklopädie der Mathematischen Wissenschaften*, which render it unnecessary to give a bibliography. The subject is one which has received the attention of the majority of the greatest mathematicians during the last two centuries and in which they have found a constant source of inspiration. Their works are generally accessible in a convenient collected form.

For the benefit of any student who wishes to supplement his reading and has no means of obtaining personal advice, the following works may be specially mentioned:

Determination of Orbits and Special Perturbations.

1. J. Bauschinger, *Bahnbestimmung der Himmelskörper.*
   (A source of fully worked numerical applications.)
2. *Publications of the Lick Observatory*, Vol. VII.
   (Contains an exposition of A. O. Leuschner's methods.)

Planetary and Lunar Theories.

3. F. Tisserand, *Traité de mécanique céleste.*
   (The most complete account of the classical theories.)
4. H. Poincaré, *Leçons de mécanique céleste.*
5. H. Poincaré, *Méthodes nouvelles de mécanique céleste.*
6. C. V. L. Charlier, *Die Mechanik des Himmels.*
7. E. W. Brown, *An introductory treatise on the lunar theory.*
   (Gives full references to all the earlier work on the subject.)

The great examples of the classical methods in the form of practical application to the theories of the planets are to be found in the works of Le Verrier (*Annales de l'Observatoire de Paris*), Newcomb (*Astronomical*

Papers of the American Ephemeris) and Hill (*Collected Works*). The most suggestive developments, apart from the researches of Poincaré, are contained in the work of H. Gyldén (*Traité analytique des orbites absolues des huit planètes principales*) and P. A. Hansen. All these works will repay careful study, but the suggestions are not to be taken in any restrictive sense.

The author of the present book has the best of reasons for acknowledging his debt to most of the writers mentioned above and to others who are not mentioned. Some of the proof sheets have been very kindly read by the Rev. P. J. Kirkby, D.Sc., late fellow of New College, Oxford. Acknowledgement is also due to the staff of the Cambridge University Press for their care in the printing. It is not to be hoped, in spite of every care, that no errors have escaped detection, and the author will be glad to have such as are found brought to his notice.

H. C. PLUMMER.

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CHAPTER I

THE LAW OF GRAVITATION

1. The foundations of dynamical Astronomy were laid by Johann Kepler at the beginning of the seventeenth century. His most important work, *Astronomia Nova* (De Motibus Stellae Martis), published in 1609, contains a profound discussion of the motion of the planet Mars, based on the observations of Tycho Brahe. In this work a real approximation to the true kinematical relations of the solar system is for the first time revealed. Kepler's main results may be summarized thus:

(*a*) The heliocentric motions of the planets (i.e. their motions relative to the Sun) take place in fixed planes passing through the actual position of the Sun.

(*b*) The area of the sector traced by the radius vector from the Sun, between any two positions of a planet in its orbit, is proportional to the time occupied in passing from one position to the other.

(*c*) The form of a planetary orbit is an ellipse, of which the Sun occupies one focus.

These laws, which were found in the first instance to hold for the Earth and for Mars, apply to the individual planets. In a later work, *Harmonices Mundi*, published in 1619, another law is given which connects the motions of the different planets together. This is:

(*d*) The square of the periodic time is proportional to the cube of the mean distance (i.e. the semi-axis major).

These deductions from observation are given here in the order in which they were discovered. The third (*c*) is generally known as Kepler's first law, the second (*b*) as his second law, and the fourth (*d*) as his third law. But the first statement is of equal importance. In the Ptolemaic system the "first inequality" of a planet, which represents its heliocentric motion, was assigned to a plane passing through the mean position of the Sun. Even in the Copernican system this "mean position" becomes the centre of the Earth's orbit, not the actual eccentric position of the Sun. In consequence no astronomer before Kepler had succeeded in representing the latitudes of the planets with even tolerable success.

2. It is undeniable that in making his discoveries Kepler was aided by a certain measure of good fortune. Thus his law of areas was in reality founded on a lucky combination of errors. In the first place it was based on the hypothesis of an eccentric circular orbit and was later adopted in the elliptic theory. In the second place Kepler supposed (a) that the time in a small arc was proportional to the radius vector, (b) that the time in a finite arc was therefore proportional to the sum of the radii vectores to all the points of the arc, (c) that this sum is represented by the area of the sector. Both (a) and (c) are erroneous, and indeed Kepler was aware that (c) was not strictly accurate. Mathematically expressed, the argument would appear thus:

$$hdt = rds, \qquad ht = \int rds = 2 \text{ (area of sector)}.$$

Both the supposed fact and the method of deduction are wrong, yet the result is right. But if it should be supposed that Kepler owed his success to good fortune it must be remembered that this fortune was simply the reward of unparalleled industry in exhausting the possibilities of every hypothesis that presented itself and in checking the value of any new principle by direct comparison with good observations. It must also be remarked that Tycho Brahe's observations were of the proper order of accuracy for Kepler's purpose, being sufficiently accurate to discriminate between true and false hypotheses and yet not so refined as to involve the problem in a maze of unmanageable detail. Another factor in Kepler's success was his knowledge of the Greek mathematicians, in particular of the works of Apollonius.

3. Kepler had no conception of the property of inertia and he was therefore unable to make any progress towards a correct dynamical view of planetary motion. It is interesting to analyze his results and to see what is implied by each of the above statements taken by itself.

According to the first statement the planets move in a field of force which is such that every trajectory is a plane curve. If we suppose that the acceleration at each point is a function of the coordinates of the point, an immediate deduction can be made as to the nature of the field of force. For let A, B be two points on a certain trajectory, and let P be a third point not in the plane of this curve. Then P can be joined to A and to B by plane trajectories. The planes in which AB, PA and PB lie meet in one point O (which may be at infinity). The acceleration at A is in the plane OAB and also in the plane OAP. Hence it is along AO. Similarly the acceleration at B is along BO, and the acceleration at P is along PO. But the point O is determined by the two points A and B. Therefore the acceleration at every point of the field is directed towards the fixed point O, and the field of force is central (or parallel). Now the planes of the orbits all pass through the Sun. Hence the Sun is the centre of the field of force in which the

planets move. For an analytical proof of the general theorem see Halphen (*Comptes Rendus*, LXXXIV, p. 944).

4. To this the second statement adds nothing with regard to the nature of the forces, and might indeed have been deduced from the first. For it tells us that

$$\int r^2 d\theta = \int (x\,dy - y\,dx) = ht$$

the Sun being the origin of coordinates and h being a constant. By differentiation we have

$$x\dot{y} - y\dot{x} = h$$

or

$$x\ddot{y} - y\ddot{x} = 0.$$

Thus $\ddot{y}/\ddot{x} = y/x$, which proves that the acceleration is towards the Sun at every point, i.e. the field of force is central. Clearly the argument might be reversed, and the law of areas deduced from the fact that the accelerations are directed towards a fixed centre, which has already been obtained from the first statement. Both this theorem and its converse are given in Newton's *Principia*, Book I, Props. I and II.

5. We shall now investigate the law of acceleration towards a fixed point under which elliptic motion is possible. In the first instance it will not be assumed that the fixed point is the focus of the ellipse. Apart from the interest of the more general result, this is the more desirable because many pairs of stars are known in the sky the components of which are observed to revolve around one another in apparent ellipses; but the plane of the motion being unknown it is only a matter of inference that either star is in the focus of the relative orbit of the other. For it is the projection of the motion on a plane perpendicular to the line of sight which is observed. Let then the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be described freely under an acceleration to the fixed point (f, g). Any point on the ellipse can be represented by $(a \cos E, b \sin E)$. The angle E which is known in analytical geometry as the eccentric angle is called in Astronomy the *eccentric anomaly* of the point. The accelerations being

$$-a \sin E \,.\, \ddot{E} - a \cos E \,.\, \dot{E}^2, \quad b \cos E \,.\, \ddot{E} - b \sin E \,.\, \dot{E}^2$$

along the two axes, we have

$$\frac{-a \sin E \,.\, \ddot{E} - a \cos E \,.\, \dot{E}^2}{a \cos E - f} = \frac{b \cos E \,.\, \ddot{E} - b \sin E \,.\, \dot{E}^2}{b \sin E - g}$$

whence

$$\frac{\ddot{E}}{\dot{E}} = \frac{ag \cos E - bf \sin E}{ab - ag \sin E - bf \cos E} \,.\, \dot{E} \qquad \text{.....................(1)}$$

This is an integrable form, giving immediately

$$\dot{E} = h(ab - ag \sin E - bf \cos E)^{-1} \quad\text{.....................(2)}$$

or

$$abE + ag \cos E - bf \sin E = h(t - t_0)$$

where h and t_0 are constants of integration. If we put $h = abn$,

$$E - \frac{f}{a}\sin E + \frac{g}{b}\cos E = n(t - t_0) \quad\text{.....................(3)}$$

and this may be considered a generalized form of what is known as Kepler's equation. By adding 2π to E it is evident that $2\pi/n = T$ is the period of a whole revolution. Kepler's form applies when the motion is about a focus of the ellipse, and can be obtained by putting $f = ae$, $g = 0$, so that

$$E - e \sin E = n(t - t_0) \quad\text{.........................(4)}$$

This equation is of fundamental importance. The point for which $E = 0$ is the nearest point on the orbit to the attracting focus and is sometimes called the *pericentre.* The corresponding time is t_0 and n is called the *mean motion.*

By (1) and (2) the components of the acceleration become

$$-a \sin E \,.\, \ddot{E} - a \cos E \,.\, \dot{E}^2 = \frac{ab(f - a\cos E)h^2}{(ab - ag\sin E - bf\cos E)^3}$$

$$b \cos E \,.\, \ddot{E} - b \sin E \,.\, \dot{E}^2 = \frac{ab(g - b\sin E)h^2}{(ab - ag\sin E - bf\cos E)^3}$$

so that the total acceleration is equal to

$$R = n^2 r\left(1 - \frac{f}{a}\cos E - \frac{g}{b}\sin E\right)^{-3} \quad\text{.....................(5)}$$

where r is the distance of the point on the orbit from (f, g).

6. Before examining this result more closely, it may be noticed that the method is quite general and may be applied to any central orbit. For if the coordinates of a point (x, y) on the curve be expressed in terms of a single parameter α, we have similarly

$$\frac{x'\ddot{\alpha} + x''\dot{\alpha}^2}{x - f} = \frac{y'\ddot{\alpha} + y''\dot{\alpha}^2}{y - g}$$

or

$$\frac{\ddot{\alpha}}{\dot{\alpha}} = -\frac{x''(y - g) - y''(x - f)}{x'(y - g) - y'(x - f)} \,.\, \dot{\alpha}$$

where x', y'... denote derivatives with respect to α, and $\dot{\alpha}$, $\ddot{\alpha}$ derivatives with respect to the time. Hence on integration,

$$\dot{\alpha} = -h\{x'(y - g) - y'(x - f)\}^{-1}$$

$$\int(x\,dy - y\,dx) - fy + gx = h(t - t_0).$$

By taking the last integration over one revolution in a closed orbit it is seen that h represents twice the area divided by the periodic time. The components of the acceleration become

$$\frac{h^2(x'y''-x''y')(x-f)}{\{x'(y-g)-y'(x-f)\}^3} \quad \text{and} \quad \frac{h^2(x'y''-x''y')(y-g)}{\{x'(y-g)-y'(x-f)\}^3}$$

and the total acceleration is therefore

$$\begin{aligned} R &= h^2 r(x'y''-x''y')\{x'(y-g)-y'(x-f)\}^{-3} \\ &= h^2 r/p^3\rho \end{aligned}$$

where ρ is the radius of curvature at the point and p is the perpendicular from (f, g) to the tangent at the point. This of course is the well-known expression for the acceleration towards the centre of attraction.

The same orbit will be described in the same periodic time under the central attraction R' to another point (f', g') if

$$R' = h^2 r'/p'^3\rho$$

that is, if

$$R'/R = p^3 r'/p'^3 r.$$

This result is equivalent to *Principia,* Book I, Prop. VII, Cor. 3.

7. We now return to equation (5) which may be written

$$R = n^2 r\left(1-\frac{fx}{a^2}-\frac{gy}{b^2}\right)^{-3} = n^2 r(q_0/q)^3 \quad \text{.................(6)}$$

where q and q_0 are the perpendiculars on the polar of (f, g) from the point (x, y) on the orbit and the centre of the ellipse respectively. Hence the ellipse represented by the general equation

$$ax^2+2hxy+by^2+2gx+2fy+1=0 \quad \text{.................(7)}$$

can be described under an acceleration directed towards the origin if the acceleration follows the law

$$R = m^2 r(1+gx+fy)^{-3}, \quad m^2 = n^2\Delta^3/C^3 \quad \text{.................(8)}$$

where Δ and C have their usual meaning for the conic (7). Conversely, if the law (8) is given, the trajectory is always a conic whatever the initial conditions may be. For (7) is a possible orbit, and f and g are determined by the law, while a, b and h are three arbitrary constants which can be chosen so as to satisfy any given conditions, such as the initial velocity given in magnitude and direction at a particular point.

There now arises the interesting question whether any other form of law besides (8) exists, for which the trajectories are always conics (Bertrand's problem). Let

$$R = m^2 r/f(x, y)$$

be such a law. Then if (7) is to be an orbit,

$$f(x, y) = (1 + gx + fy)^3$$

must be satisfied by the coordinates of every point on (7), i.e. this equation must be equivalent to (7). But (7) can be written in either of the forms

$$1 + gx + fy = \tfrac{1}{2}(1 - ax^2 - 2hxy - by^2)$$
$$(1 + gx + fy)^2 = (g^2 - a)x^2 + 2(fg - h)xy + (f^2 - b)y^2$$

and clearly in no other way which does not introduce a greater number of independent constants on the right-hand side. The first of these forms gives an expression for $f(x, y)$ which is (like an infinite number of others) compatible with (7), but only under restricted conditions. For it fixes the constants a, b and h and leaves only f and g arbitrary; and these are not in general sufficient in number to satisfy the initial conditions. On the other hand, the second form gives an expression for the acceleration which may be written

$$R = m^2 r(\alpha x^2 + 2\beta xy + \gamma y^2)^{-\frac{3}{2}} \quad \text{.....................(9)}$$

This only requires the constants in (7) to satisfy the two relations

$$\frac{g^2 - a}{\alpha} = \frac{fg - h}{\beta} = \frac{f^2 - b}{\gamma}$$

and thus three other relations can be satisfied which are required by the initial conditions. Hence motion under a central acceleration given by (9) is always in a conic which by the two relations found touches the lines (real or imaginary)

$$\alpha x^2 + 2\beta xy + \gamma y^2 = 0.$$

The laws (8) and (9) are the only ones under which a conic is always described in a given plane whatever the initial conditions may be. Their character was first established by Darboux and by Halphen (*Comptes Rendus*, LXXXIV, pp. 760, 936 and 939).

8. A point on a central orbit at which the motion is at right angles to the radius vector is called an *apse*. At such a point $\dfrac{dr}{d\theta} = 0$ and the radius vector is in general either a maximum or a minimum. Since the motion is reversible the radius vector to an apse is an axis of symmetry in the orbit and the next apsidal distances on either side are equal. There can be therefore only two distinct apsidal distances recurring alternately and the angle between any two consecutive apses is constant and is called the apsidal angle.

The differential equation of a central orbit is known to be

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

where $u = 1/r$ and P is the force to the centre. If we write $P = u^2 U$ the radius of a circular orbit is given by $u = U/h^2$. Let the circular orbit be slightly disturbed, so that we may write $u + x$ instead of u, where u is constant and x is so small that only the first power of x need be retained. Then

$$\frac{d^2x}{d\theta^2} + x = \frac{U'}{h^2}x = \frac{uU'}{U}x, \quad U' = \frac{dU}{du}.$$

If we put

$$1 - uU'/U = m^2$$

the equation becomes

$$\frac{d^2x}{d\theta^2} + m^2x = 0$$

and the solution is

$$x = a \cos m(\theta - \theta_0).$$

The apsidal angle is therefore

$$K = \pi/m = \pi(1 - uU'/U)^{-\frac{1}{2}} \quad \text{......................(10)}$$

For example, if $P = \mu r^p$, $U = \mu u^{-p-2}$ and

$$K = \pi(3 + p)^{-\frac{1}{2}}.$$

This result is given in the *Principia*, Book I, Prop. XLV, Ex. 2.

9. Let us push the approximation further in order to see, if possible, under what conditions the apsidal angle remains unchanged by a higher order of the increment x. The equation of the disturbed circular orbit becomes

$$\frac{d^2x}{d\theta^2} + m^2x = \frac{u}{U}\left(\tfrac{1}{2}U''x^2 + \tfrac{1}{6}U'''x^3\right) \quad \text{..................(11)}$$

and we assume a solution

$$x = a_0 + a_1 \cos m\theta + a_2 \cos 2m\theta + a_3 \cos 3m\theta.$$

If a_1 is of the first order, a_0 and a_2 must be of the second order at least, and it will become clear that a_3 is of the third order. Hence

$$x^2 = \tfrac{1}{2}a_1^2 + (2a_0a_1 + a_1a_2)\cos m\theta + \tfrac{1}{2}a_1^2 \cos 2m\theta + a_1a_2 \cos 3m\theta$$
$$x^3 = \tfrac{3}{4}a_1^3 \cos m\theta + \tfrac{1}{4}a_1^3 \cos 3m\theta.$$

All terms of order higher than the third have been omitted and products of the cosines have been changed into simple cosines of the multiple angles. We now substitute in (11) and equate coefficients. Thus

$$m^2a_0 = \frac{1}{4}\cdot\frac{uU''}{U}\cdot a_1^2$$

$$0 = \frac{1}{2}\cdot\frac{uU''}{U}\cdot(2a_0a_1 + a_1a_2) + \frac{1}{8}\cdot\frac{uU'''}{U}\cdot a_1^3$$

$$-3m^2a_2 = \frac{1}{4}\cdot\frac{uU''}{U}\cdot a_1^2$$

$$-8m^2a_3 = \frac{1}{2}\cdot\frac{uU''}{U}\cdot a_1a_2 + \frac{1}{24}\cdot\frac{uU'''}{U}\cdot a_1^3.$$

The last of these equations confirms the statement that a_3 is of the third order, but will not be needed here. The first three after the elimination of a_0 and a_2 give

$$0=\left\{\frac{1}{2}\frac{uU''}{m^2U}\cdot\frac{5}{12}\frac{uU''}{U}+\frac{1}{8}\frac{uU'''}{U}\right\}a_1^3$$

or

$$5uU''^2+3U'''(U-uU')=0 \quad\text{.....................(12)}$$

This equation expresses a necessary condition which must be satisfied if the apsidal angle is to remain constant when the displacement from a circular orbit is considered finite.

10. Let us consider any closed orbit to be determined by a central acceleration under a finite range of initial velocities. The number of apses in a complete orbit must be finite and (10) shows that m must be a commensurable number. It must be a constant therefore, for otherwise it would change discontinuously as u changes continuously. Hence

$$m^2=1-uU'/U$$

is an equation giving the form of U, and the solution is

$$U=ku^{1-m^2}.$$

But if all the orbits are to be re-entrant, so that K is constant, the equation (12) must also be satisfied. Hence substituting the form just found, we have

$$5m^4(1-m^2)^2+3m^4(1-m^4)=0$$

or

$$2m^4(4-m^2)(1-m^2)=0.$$

Since K is finite, m is not zero and we have

$$1-m^2=0 \quad\text{or}\quad 1-m^2=-3$$

giving

$$U=k \qquad\text{or}\qquad U=ku^{-3}$$

and

$$R=k/r^2 \qquad\text{or}\qquad R=kr.$$

Thus we have Bertrand's remarkable theorem (*Comptes Rendus*, LXXVII, p. 849) that these are the only laws, expressible as functions of the distance, which always give rise to closed orbits whatever the initial circumstances may be (within a certain range). In these two cases $m=1$ or 2 and the apsidal angle $K=\pi$ or $\frac{1}{2}\pi$.

11. The results obtained can now be brought together. According to Kepler's law the planetary orbits are ellipses with the centre of attraction, the Sun, situated in one focus. The polar of the focus being the corresponding directrix, we have in (6) $q_0=a/e$ and $q=r/e$, so that the acceleration towards the Sun is

$$R=n^2a^3/r^2 \quad\text{.................................(13)}$$

When the centre of attraction is an arbitrary point and it is merely known that the orbits are ellipses, the acceleration towards the centre must

follow one of the two laws expressed by (8) and (9). These are not in general simple functions of the distance and it is only by induction that we should infer from the apparent orbits of double stars that these bodies obey the law given by (13). But the law (8) provides a simple function of the distance, $R = m^2 r$, when $f = g = 0$, in which case the centres of all possible orbits are at the origin, i.e. coincide with the centre of attraction. Similarly the law (9) provides a simple function of the distance, $R = m^2/r^2$, when $\alpha = \gamma$ and $\beta = 0$. In this case every orbit touches the lines $x^2 + y^2 = 0$, showing that the centre of attraction at the origin is the focus for every path. These are the only two laws of central acceleration which give rise to elliptic orbits in general and can be expressed in simple terms of the distance. But we have also seen that the same restriction is imposed when it is merely required that the paths shall be plane closed curves of any kind. It is moreover obvious that the law of the direct distance, which makes the attraction of a distant body more effective than that of a near one, cannot be the law of nature. The only alternative is that the acceleration varies inversely as the square of the distance, and this law can therefore be based upon these simple suppositions: (*a*) the planets describe closed paths in planes passing through the Sun, (*b*) the centripetal acceleration towards the Sun, required by (*a*), is a simple function of the distance and does not become infinite when the distance is infinite.

12. We have now to consider Kepler's law connecting the periodic times of the planets with their mean distances from the Sun. This states that T^2 varies as a^3. But $T = 2\pi/n$, so that n^2a^3 is constant for all the planets. Hence by (13) the acceleration of each planet towards the Sun is μ/r^2 where μ is constant. The force of attraction acting on a planet is therefore $m\mu/r^2$ where m is the mass of the planet. And observation shows that the same form of law holds for the satellites of any planet, e.g. the satellites of Jupiter. Thus not only does the Sun attract the planets but the planets themselves appear to attract their satellites in the same way. It is but natural to suppose that the forces of attraction in either case arise from an inherent property of matter, and that a stress exists between the Sun and a planet, or between a planet and its satellite. Action and reaction being equal and opposite, we must suppose the force proportional not only to the mass of the attracted body but equally to the mass of the attracting body. We are thus led to Newton's law of gravitation that the mutual attraction between two masses m, m' at a distance r apart is measured by

$$Gmm'/r^2$$

where G is an absolute constant, independent of the masses or their distance. It must be noticed that the law has been arrived at from the consideration of cases in which the dimensions of the bodies are small in comparison with the distances separating them. But since the action in these cases is proportional

to the total masses, it is to be supposed that it applies to the individual elements of the matter composing them. This is the true form of the law of universal gravitation. When it is a question of bodies whose dimensions are not negligible in relation to the distances of surrounding bodies, a modification of the simple statement must be expected. The examination of all consequences of the law of gravitation, including a comparison with the results of observation, practically constitutes the complete function of dynamical Astronomy.

13. Since the Earth possesses only one satellite, it is impossible to verify Kepler's third law in our own system. But it is of historic interest to calculate from the observed motion of the Moon the acceleration towards the centre of the Earth which a body would have at the Earth's surface. The Moon's sidereal period is $27^d\ 7^h\ 43^m\ 11^s{\cdot}5$ or $2360591{\cdot}5$ secs. Let a be the Moon's mean distance and b the radius of the Earth. The required acceleration is

$$\frac{n^2a^3}{b^2} = \frac{4\pi^2}{T^2} \cdot \left(\frac{a}{b}\right)^3 . b.$$

The ratio a/b is $60{\cdot}2745$ and b may be taken to be $6{\cdot}378 \times 10^8$ cm. The result of substituting these numbers is to give for the acceleration 989 cm./sec.^{2} In point of fact the acceleration of a body at the Earth's surface is in the mean $g = 981$ cm./sec.^{2} But the discrepancy is not surprising. The Moon describes its orbit not only under the attraction of the Earth but also under the disturbing influence of the Sun. Moreover g is a variable quantity over the Earth's surface, owing to the Earth's rotation and figure. The above calculation is altogether too rough to give really comparable results. But it suffices to show that the quantity is quite of the same order as g, and to this extent supports the identification of the force which retains the Moon in its orbit with that which in the case of terrestrial objects is known as weight. As stated, the point is of historical interest because it presented a difficulty to Newton who was long misled by adopting erroneous numerical data.

14. The numerical value of the constant G depends upon the units adopted. Its dimensions are given by

$$G . M^2L^{-2} = MLT^{-2}$$

or

$$G = M^{-1}L^3T^{-2}.$$

In C.G.S. units it is the force between two particles each of 1 gramme placed 1 cm. apart. The first determination of the force in absolute units by a laboratory experiment was made by Cavendish. Several determinations have since been made, of which perhaps the two best, those of C. V. Boys and K. Braun, agree in giving

$$G = 6{\cdot}658 \times 10^{-8}$$

corresponding to $5{\cdot}527$ for the mean density of the Earth and $5{\cdot}985 \times 10^{27}$ gr. for the total mass of the Earth.

CHAPTER II

INTRODUCTORY PROPOSITIONS

15. As we have seen, the simple facts of observation lead us to assume that between two particles of masses m and m' situated at the points $P(x, y, z)$ and $P'(x', y', z')$ there exists a force Gmm'/r^2, where r is the distance PP'. Now the direction cosines of PP' are

$$\frac{x'-x}{r}, \quad \frac{y'-y}{r}, \quad \frac{z'-z}{r}$$

and hence the components of the force acting on the particle m are

$$Gmm'\frac{x'-x}{r^3}, \quad Gmm'\frac{y'-y}{r^3}, \quad Gmm'\frac{z'-z}{r^3}$$

or

$$-\frac{\partial U}{\partial x}, \quad -\frac{\partial U}{\partial y}, \quad -\frac{\partial U}{\partial z}$$

where

$$U = -Gmm'/r.$$

If m is attracted not by a single particle m' but by any number typified by m_i at (x_i, y_i, z_i) the components of the total force are similarly

$$-\frac{\partial U}{\partial x}, \quad -\frac{\partial U}{\partial y}, \quad -\frac{\partial U}{\partial z}$$

where

$$U = -Gm\sum_i m_i/r_i.$$

It is evident that U is the work which the system of attracting particles will do if the particle m is moved from its actual position by any path to some standard position, except for a constant; it is the potential energy of m due to its position relative to the attracting system. If we put

$$V = G\sum_i m_i/r_i, \quad U = -mV$$

V is called the potential of the attracting system at the point P. When the potential is known it is evident that the components of the attraction can be easily calculated.

16. The case of a homogeneous spherical shell is of elementary importance. Let m be the mass per unit area, a the radius and r the distance of the point P from the centre. If O is the centre of the sphere, two cones with semi-vertical angles ϕ and $\phi + d\phi$, each having its vertex at O and OP as its axis, will contain between them an annulus on the surface of the sphere. The potential of this annulus at P is

$$dV = Gm \,.\, 2\pi a \sin\phi \,.\, ad\phi/\rho$$

where

$$\rho^2 = r^2 + a^2 - 2ra\cos\phi$$

or

$$\rho d\rho = ra \sin\phi \,.\, d\phi$$

so that

$$dV = Gm \,.\, 2\pi a\, d\rho/r.$$

Hence

$$V = 2\pi Gma\,(\rho_2 - \rho_1)/r$$

where ρ_2 and ρ_1 are the values of ρ at the ends of the diameter through P. These values are

$$\rho_2 = r + a, \quad \rho_1 = |\,r - a\,|.$$

If $r > a$, $\rho_1 = r - a$ and $\rho_2 - \rho_1 = 2a$; if $r < a$, $\rho_1 = a - r$ and $\rho_2 - \rho_1 = 2r$. Also the whole mass of the shell is $M = 4\pi ma^2$. Hence when P is a point outside the shell

$$V = GM/r$$

or the potential and the forces derived from it are the same as if the whole mass of the shell were concentrated at the centre. On the other hand, when P is a point inside the shell,

$$V = GM/a$$

or the potential is constant and the forces derived from it are zero.

17. From this elementary proposition follow immediately two corollaries:

(1) A sphere of uniform density, or one composed of concentric strata of uniform density, may be treated, so far as its action at an external point is concerned, as equivalent to a single particle of equal mass placed at its centre.

(2) For a point within such a sphere, the sphere may be divided into two parts by the concentric sphere passing through the point. The outer part is inoperative and may be ignored, while the inner may be replaced by a particle of equal mass situated at the centre.

The heavenly bodies are for the most part approximately spherical in shape, and though not uniform in density their concentric strata are in general fairly homogeneous. They may therefore be treated in most cases, as regards their action on other bodies, as simple particles.

The motion of a body within a sphere may be illustrated by the motion of a meteor within a spherical swarm, or of a star in a spherical cluster. If

the swarm fills a sphere uniformly the mass operative at any point varies as the cube of the distance from the centre. Hence the effective force towards the centre varies directly as the distance. As another example it may be proved that if the density of a globular cluster varies as $(1+r^2)^{-\frac{5}{2}}$, r being the distance from the centre, each star moves under a central attraction varying as $r(1+r^2)^{-\frac{3}{2}}$.

18. An approximate expression can be found for the potential of a body of any shape at a distant point. Let the origin of coordinates, O, be taken at the centre of gravity of the body and the axis of x be drawn through the point P, the distance OP being r. Let dm be an element of mass at the point (x, y, z). The corresponding element of the potential at P is

$$dV = \frac{G\,dm}{\{(r-x)^2+y^2+z^2\}^{\frac{1}{2}}} = \frac{G\,dm}{(r^2-2rx+\rho^2)^{\frac{1}{2}}}$$

$$= \frac{G\,dm}{r}\left(1 - 2\frac{\rho}{r}\cdot\frac{x}{\rho} + \frac{\rho^2}{r^2}\right)^{-\frac{1}{2}}$$

$$= \frac{G\,dm}{r}\left\{1 + \frac{\rho}{r}P_1\left(\frac{x}{\rho}\right) + \left(\frac{\rho}{r}\right)^2 P_2\left(\frac{x}{\rho}\right) + \dots\right\},$$

where $P_1, P_2, \dots$ are the functions known as Legendre's polynomials.

The first terms are easily obtained by expansion in the ordinary way, and we have

$$P_1\left(\frac{x}{\rho}\right) = \frac{x}{\rho},\quad P_2\left(\frac{x}{\rho}\right) = \frac{3x^2-\rho^2}{2\rho^2}.$$

Hence if the expansion is not carried to terms beyond the second order,

$$V = G\int \frac{dm}{r}\left(1 + \frac{x}{r} + \frac{3x^2-\rho^2}{2r^2}\right).$$

But if A, B, C are the principal moments of inertia at O, and I is the moment of inertia about Ox, since ρ^2 has been written for $x^2+y^2+z^2$,

$$A+B+C = \int 2\rho^2 dm,\quad I = \int (\rho^2 - x^2)\,dm$$

and since O is the centre of gravity,

$$\int x\,dm = 0.$$

Hence

$$V = \frac{Gm}{r} + \frac{G}{2r^3}(A+B+C-3I)$$

and we see that the potential of the body at P differs from the potential of a particle of equal total mass placed at the centre of gravity by a quantity depending only on $1/r^3$. Except in a few cases this quantity is negligible

in astronomical problems not only by reason of the great distances which separate the heavenly bodies in comparison with their linear dimensions, but because they possess in general a symmetry of form which makes $A+B+C-3I$ itself a small quantity.

19. We see then that in general a system of n bodies of finite dimensions can be replaced by a system of n small particles of equal masses occupying the positions of their centres of gravity. The total potential energy of the system is

$$U = -G\Sigma m_i m_j / r_{ij}$$

where m_i, m_j are two of the masses and r_{ij} their distance apart. For if we start with any one of the particles this sum, which consists of $\frac{1}{2}n(n-1)$ terms, represents the potential energy of a second in the presence of the first, of a third in the presence of these two, and so on. The equations of motion are $3n$ in number and, according to § 15, of the form

$$m_i\ddot{x}_i = -\frac{\partial U}{\partial x_i}, \quad m_i\ddot{y}_i = -\frac{\partial U}{\partial y_i}, \quad m_i\ddot{z}_i = -\frac{\partial U}{\partial z_i}.$$

Now

$$\sum_i \frac{\partial U}{\partial x_i} = \sum_i\sum_j m_i m_j \frac{x_i - x_j}{r_{ij}^3} = 0, \qquad (i \neq j).$$

Hence

$$\Sigma m_i\ddot{x}_i = \Sigma m_i\ddot{y}_i = \Sigma m_i\ddot{z}_i = 0$$

or

$$\Sigma m_i\dot{x}_i = a_1, \ \Sigma m_i\dot{y}_i = a_2, \ \Sigma m_i\dot{z}_i = a_3$$

and

$$\Sigma m_i x_i = \bar{x}\Sigma m_i = a_1 t + b_1$$
$$\Sigma m_i y_i = \bar{y}\Sigma m_i = a_2 t + b_2$$
$$\Sigma m_i z_i = \bar{z}\Sigma m_i = a_3 t + b_3$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centre of gravity of the system. Thus we have the six integrals corresponding to the fact that the centre of gravity moves with uniform velocity in a certain direction.

Again, we have

$$\sum_i \left(y_i \frac{\partial U}{\partial z_i} - z_i \frac{\partial U}{\partial y_i}\right) = \sum_i\sum_j m_i m_j \left\{y_i \frac{z_i - z_j}{r_{ij}^3} - z_i \frac{y_i - y_j}{r_{ij}^3}\right\}$$

$$= \sum_i\sum_j \frac{m_i m_j}{r_{ij}^3}(-y_i z_j + y_j z_i) = 0, \qquad (i \neq j).$$

Hence

$$\Sigma m_i(y_i\ddot{z}_i - z_i\ddot{y}_i) = 0$$

or

$$\Sigma m_i(y_i\dot{z}_i - z_i\dot{y}_i) = c_1$$

and similarly

$$\Sigma m_i(z_i\dot{x}_i - x_i\dot{z}_i) = c_2$$

$$\Sigma m_i(x_i\dot{y}_i - y_i\dot{x}_i) = c_3.$$

These are called the three integrals of area and express the fact that the sum of the areas described by the radius vector to each mass, each multiplied by that mass and projected on any given plane, is constant. They also show that the total angular momentum of the system about any fixed axis is constant.

Finally we have

$$\begin{aligned}\sum_i m_i(\dot{x}_i\ddot{x}_i+\dot{y}_i\ddot{y}_i+\dot{z}_i\ddot{z}_i) &= -\sum_i\left(\dot{x}_i\frac{\partial U}{\partial x_i}+\dot{y}_i\frac{\partial U}{\partial y_i}+\dot{z}_i\frac{\partial U}{\partial z_i}\right)\\ &= -dU/dt\end{aligned}$$

whence, on integration,

$$\tfrac{1}{2}\sum_i m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)=h-U$$

where h is constant. This is the integral of energy.

There are then in all ten general integrals for the motion of a system of particles moving under their mutual attractions: and it is known that no others exist under certain limitations of analytical form (Bruns and Poincaré). They are in fact simply those which apply in virtue of the absence of external forces acting on the system.

20. Let the centre of gravity $(\bar{x}, \bar{y}, \bar{z})$ of the system be now taken as the origin of coordinates. If (ξ_i, η_i, ζ_i) are the new coordinates of m_i,

$$x_i=\bar{x}+\xi_i,\quad y_i=\bar{y}+\eta_i,\quad z_i=\bar{z}+\zeta_i$$

and

$$\Sigma m_i\xi_i=\Sigma m_i\eta_i=\Sigma m_i\zeta_i=0.$$

The equations of motion become

$$m_i\ddot{\xi}_i=-\frac{\partial U}{\partial \xi_i},\quad m_i\ddot{\eta}_i=-\frac{\partial U}{\partial \eta_i},\quad m_i\ddot{\zeta}_i=-\frac{\partial U}{\partial \zeta_i}$$

where U is the same as before, but r_{ij} is now given by

$$r_{ij}^2=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2+(\zeta_i-\zeta_j)^2.$$

For the integrals of area we have

$$\begin{aligned}c_1 &= \Sigma m_i(y_i\dot{z}_i-z_i\dot{y}_i)\\ &= \Sigma m_i\{(\bar{y}+\eta_i)(\dot{\bar{z}}+\dot{\zeta}_i)-(\bar{z}+\zeta_i)(\dot{\bar{y}}+\dot{\eta}_i)\}\\ &= \Sigma m_i(\eta_i\dot{\zeta}_i-\zeta_i\dot{\eta}_i)+(\bar{y}\dot{\bar{z}}-\bar{z}\dot{\bar{y}})\,\Sigma m_i\end{aligned}$$

(since $\Sigma m_i\eta_i=\Sigma m_i\zeta_i=\Sigma m_i\dot{\eta}_i=\Sigma m_i\dot{\zeta}_i=0$)

$$= \Sigma m_i(\eta_i\dot{\zeta}_i-\zeta_i\dot{\eta}_i)+(a_3b_2-a_2b_3)/\Sigma m_i$$

or

$$\Sigma m_i(\eta_i\dot{\zeta}_i-\zeta_i\dot{\eta}_i)=c_1+(a_2b_3-a_3b_2)/\Sigma m_i=c_1'$$

and similarly

$$\Sigma m_i(\zeta_i\dot{\xi}_i-\xi_i\dot{\zeta}_i)=c_2+(a_3b_1-a_1b_3)/\Sigma m_i=c_2'$$

$$\Sigma m_i(\xi_i\dot{\eta}_i-\eta_i\dot{\xi}_i)=c_3+(a_1b_2-a_2b_1)/\Sigma m_i=c_3'.$$

The integral of energy becomes

$$h - U = \tfrac{1}{2}\Sigma m_i \{(\dot{\bar{x}} + \dot{\xi}_i)^2 + (\dot{\bar{y}} + \dot{\eta}_i)^2 + (\dot{\bar{z}} + \dot{\zeta}_i)^2\}$$

$$= \tfrac{1}{2}\Sigma m_i (\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) + \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2)/\Sigma m_i$$

or

$$\tfrac{1}{2}\Sigma m_i (\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) = h' - U$$

where

$$h' = h - \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2)/\Sigma m_i.$$

21. An interesting equation involving the mutual distances of the masses can be deduced. We have

$$2 \sum_{i,j} m_i m_j (\xi_i - \xi_j)^2 = \sum_i \sum_j m_i m_j (\xi_i^2 + \xi_j^2 - 2\xi_i \xi_j)$$

$$= \Sigma m_i \xi_i^2 \,.\, \Sigma m_j + \Sigma m_i \,.\, \Sigma m_j \xi_j^2 - 2\Sigma m_i \xi_i \,.\, \Sigma m_j \xi_j$$

$$= 2\Sigma m_i \,.\, \Sigma m_i \xi_i^2$$

with similar equations for the other coordinates. Hence

$$\Sigma m_i m_j r_{ij}^2 = \Sigma m_i \,.\, \Sigma m_i (\xi_i^2 + \eta_i^2 + \zeta_i^2).$$

It follows that

$$\frac{d^2}{dt^2}(\Sigma m_i m_j r_{ij}^2)/\Sigma m_i = 2\frac{d}{dt}\{\Sigma m_i (\xi_i \dot{\xi}_i + \eta_i \dot{\eta}_i + \zeta_i \dot{\zeta}_i)\}$$

$$= 2\Sigma m_i (\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) - 2\Sigma \left(\xi_i \frac{\partial U}{\partial \xi_i} + \eta_i \frac{\partial U}{\partial \eta_i} + \zeta_i \frac{\partial U}{\partial \zeta_i}\right)$$

$$= 4(h' - U) + 2U = 4h' - 2U$$

since U is a homogeneous function of the coordinates of degree -1. The form of the result is due to Jacobi. Now U is essentially negative. Hence if h' be positive the second derivative of $\Sigma m_i m_j r_{ij}^2$ will be always positive and the first derivative will increase indefinitely with the time. Thus the first derivative, even if negative initially, will become positive after a certain time and therefore $\Sigma m_i m_j r_{ij}^2$ will increase without limit. This means that at least one of the distances will tend to become infinite. We see therefore that a necessary (but not sufficient) condition for the stability of the system is that h' must be negative.

22. The angular momenta whose constant values are c_1, c_2, c_3 are the projections on the coordinate planes of a single quantity. They are therefore the components of a vector which represents the resultant angular momentum about the axis

$$x/c_1 = y/c_2 = z/c_3 \quad \text{..............................(1)}$$

For this axis, which is fixed in space, the angular momentum is a maximum. The plane through the origin O which is perpendicular to this axis and therefore fixed is called the *invariable plane* at O. About any line through O in this plane the angular momentum is zero, and about any line through O

making an angle θ with the invariable axis (1) the angular momentum is $\sqrt{(c_1^2 + c_2^2 + c_3^2)} \cos\theta$. The position of the invariable plane is dependent on the position of the chosen origin of reference.

Here we have considered the angular momentum as arising purely from the translational motions of the bodies treated as particles. In reality the total angular momentum of the system includes also that part which arises from the rotations of the bodies about their axes. This part ~~itself~~ is constant if the system consists of unconnected, rigid, spherical bodies whose concentric layers are homogeneous. Under these conditions the invariable plane at a point, as determined by the translational motions of the system alone, remains permanently fixed. The conditions hold very approximately in a planetary system. But precessional movements and the effects of tidal friction cause an interchange between the rotational and translational parts of the angular momentum, without disturbing the total amount, and to this extent affect the position of the astronomical invariable plane as defined above.

The centre of gravity of the system may be taken instead of an origin fixed in space. The invariable plane is then

$$c_1'\xi + c_2'\eta + c_3'\zeta = 0 \quad\text{..............................(2)}$$

and this is the invariable plane of Laplace. Its permanent fixity is subject to the qualifications just mentioned.

A simple proposition applies to the motion of two bodies, namely that the planes through a fixed point O and containing the tangents to the paths of the two bodies intersect the invariable plane at O in one line. This is easily seen to be true. For the first plane passes through the origin, the position of the first body (x_1, y_1, z_1) and the consecutive point on its path $(x_1 + \dot{x}_1 dt,\ y_1 + \dot{y}_1 dt,\ z_1 + \dot{z}_1 dt)$. Hence its equation is

$$x(y_1\dot{z}_1 - \dot{y}_1 z_1) + y(z_1\dot{x}_1 - \dot{z}_1 x_1) + z(x_1\dot{y}_1 - \dot{x}_1 y_1) = 0.$$

Similarly the equation of the second plane is

$$x(y_2\dot{z}_2 - \dot{y}_2 z_2) + y(z_2\dot{x}_2 - \dot{z}_2 x_2) + z(x_2\dot{y}_2 - \dot{x}_2 y_2) = 0.$$

The equations of these planes together with that of the invariable plane may therefore be written

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad m_1\alpha_1 + m_2\alpha_2 = 0$$

and these evidently meet in a common line of intersection.

23. When we deal with the motions in the solar system it is convenient to refer them to the centre of the Sun as origin. Let M be the mass of the Sun, m the mass of the planet specially considered and let there be n other

planets, of which the typical mass is m_i. Then the total potential energy of the system is

$$U = -\left(\Sigma \frac{m_i m_j}{r_{ij}} + M\Sigma \frac{m_i}{\rho_i} + m\Sigma \frac{m_i}{\Delta_i} + \frac{mM}{r}\right) G$$

where ρ_i is the distance of m_i from the Sun, Δ_i the distance of m_i from m and r the distance of m from the Sun, so that

$$\begin{aligned}
r_{ij}^2 &= (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \\
\rho_i^2 &= (x_i - X)^2 + (y_i - Y)^2 + (z_i - Z)^2 \\
\Delta_i^2 &= (x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2 \\
r^2 &= (x - X)^2 + (y - Y)^2 + (z - Z)^2.
\end{aligned}$$

The equations of motion of the Sun are

$$M\ddot{X} = -\frac{\partial U}{\partial X}, \quad M\ddot{Y} = -\frac{\partial U}{\partial Y}, \quad M\ddot{Z} = -\frac{\partial U}{\partial Z}$$

and of the planet considered

$$m\ddot{x} = -\frac{\partial U}{\partial x}, \quad m\ddot{y} = -\frac{\partial U}{\partial y}, \quad m\ddot{z} = -\frac{\partial U}{\partial z}.$$

If (ξ, η, ζ) are the relative coordinates of the planet,

$$x = X + \xi, \quad y = Y + \eta, \quad z = Z + \zeta.$$

Hence, if (ξ_i, η_i, ζ_i) are the coordinates of m_i relative to the Sun,

$$\begin{aligned}
\ddot{\xi} &= -\frac{1}{m}\frac{\partial U}{\partial x} + \frac{1}{M}\frac{\partial U}{\partial X} \\
&= \left\{-\Sigma \frac{m_i (x - x_i)}{\Delta_i^3} - \frac{M(x - X)}{r^3} + \Sigma \frac{m_i (X - x_i)}{\rho_i^3} + \frac{m(X - x)}{r^3}\right\} G \\
&= \left\{-\frac{(m + M)\xi}{r^3} - \Sigma \frac{m_i(\xi - \xi_i)}{\Delta_i^3} - \Sigma \frac{m_i \xi_i}{\rho_i^3}\right\} G.
\end{aligned}$$

If then we put

$$R = G\left\{\Sigma \frac{m_i}{\Delta_i} - \Sigma \frac{m_i}{\rho_i^3}(\xi\xi_i + \eta\eta_i + \zeta\zeta_i)\right\} \quad \text{.................(3)}$$

we have for the equations of relative motion

$$\ddot{\xi} = -(m + M)G \cdot \frac{\xi}{r^3} + \frac{\partial R}{\partial \xi} \quad \text{.........................(4)}$$

and similarly

$$\ddot{\eta} = -(m + M)G \cdot \frac{\eta}{r^3} + \frac{\partial R}{\partial \eta} \quad \text{.........................(5)}$$

$$\ddot{\zeta} = -(m + M)G \cdot \frac{\zeta}{r^3} + \frac{\partial R}{\partial \zeta} \quad \text{.........................(6)}$$

The function R is called the *disturbing function.* When, as in the solar system, the masses of the planets are small in comparison with that of the central body, M, we see that the forces derived from this function are small in comparison with the attraction of M. Indeed a first approximation to the motion of the planet considered, which may now be called the disturbed planet, is obtained by putting $R=0$.

24. A double star, or system of two stars physically connected and at the same time isolated from external influences, may be considered to present a case of the problem of two bodies. In the solar system the disturbing effect of the other planets is always operating. Since, however, this effect is small in comparison with the attraction of the Sun it is useful to neglect R and to consider the orbit which a particular planet would have if at a given instant the disturbing forces were removed and the planet continued to move as part of the system formed by itself and the Sun alone, its velocity in direction and amount at the given instant being that which it actually possesses. Such an orbit is called the *osculating orbit* corresponding to the given instant. The actual orbit from the beginning will depart more and more from the osculating orbit, but for a short interval of time the divergence between the two will be so small that an accurate ephemeris can be calculated from the elements of the osculating orbit. The usefulness of the conception of the osculating orbit goes much deeper than this, as will appear later.

Now the equations (4) to (6) show that in the problem of two bodies, since $R=0$, the relative motion is that which is determined by an acceleration $(m+M)\,G/r^2$ towards the body M which is considered fixed. But by § 11 (13) a law of this form leads to an elliptic orbit with mean distance a and periodic time T, where

$$nT = 2\pi, \quad n^2a^3 = (m+M)\,G.$$

We can now introduce the usual system of astronomical units. Provisionally they are taken to be:

Unit of time: one mean solar day.

Unit of length: the Earth's mean distance from the Sun.

Unit of mass: the Sun's mass.

Corresponding to this system G is replaced by the constant k^2, so that

$$k = 2\pi/(1+m)^{\frac{1}{2}}\,T$$

which differs little from the Earth's mean motion. Here T is the sidereal year expressed in mean solar days and m is the mass of the Earth expressed as a fraction of that of the Sun. The numerical values adopted by Gauss were:

$$T = 365\cdot 256\,3835$$

$$m = 1/354\,710$$

which lead to

$$k = 0{\cdot}017\,202\,098\,95, \quad \log k = 8{\cdot}235\,581\,4414 - 10.$$

It may be useful to add that

$$180^\circ . k/\pi = 3548''{\cdot}18761, \quad \log(180^\circ . k/\pi) = 3{\cdot}550\,006\,5746$$

which differs little from the Earth's daily mean motion expressed in seconds.

The number k is called the Gaussian constant. The numerical values of m and T on which it is based are no longer considered accurate. Nevertheless it would cause great practical inconvenience to adjust the value of k to more modern values which themselves could not be regarded as final. Hence it is agreed to adopt the above value of k as a definite, arbitrary constant and to recognize that the corresponding unit of length is only an approximation to the Earth's mean distance from the Sun. According to Newcomb the logarithm of this distance is 0·000 000 013.

It is also possible to put the constant $k = 1$ by adopting as the unit of time $1/k = 58{\cdot}132\,44087$ mean solar days.

For brevity we may often put

$$\mu = k^2(1 + m) = n^2a^3$$

in the case of a planetary orbit, and for a double star

$$\mu = k^2(M + m) = n^2a^3$$

where M, m are the masses of the two components when the mass of the Sun is taken as unity.

CHAPTER III

MOTION UNDER A CENTRAL ATTRACTION

25. If the attraction of the Sun alone is considered, the relative motion of any other body of spherical shape is conditioned by the central acceleration μr^{-2}, μ being a constant the value of which has been explained. The equations of motion expressed in polar coordinates are:

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

The latter equation gives immediately

$$r^2\dot{\theta} = h$$

where h is the constant of areas. Let v be the velocity in the orbit, P the perpendicular from the origin on the tangent and ψ the angle which the tangent makes with the radius vector. Then

$$\frac{r\dot{\theta}}{v} = \sin\psi = \frac{P}{r}$$

so that

$$Pv = r^2\dot{\theta} = h$$

or the velocity is inversely proportional to P. The result of eliminating θ from the equations of motion is

$$\ddot{r} = h^2/r^3 - \mu/r^2$$

whence

$$\dot{r}^2 = 2\mu/r - h^2/r^2 + c \qquad \text{.....(1)}$$

and from these again

$$\frac{d^2}{dt^2}(r^2) = 2(r\ddot{r} + \dot{r}^2) = 2\mu/r + 2c.$$

The equation of energy is

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = 2\mu/r + c \qquad \text{.....(2)}$$

The geometrical meaning of the constant c has yet to be found.

26. From the second equation of motion

$$\frac{d}{dt} = hu^2 \frac{d}{d\theta}$$

where $u = 1/r$. Hence the first equation of motion becomes

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0$$

the integral of which is

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \gamma)\} \quad \text{.........................(3)}$$

where e and γ are the two constants of integration. But this is the polar equation of a conic section of which the eccentricity is e and the focus is at the origin. The semi-latus rectum in this connexion is more usually called the *parameter* and denoting it by p we have

$$p = h^2/\mu \quad \text{or} \quad h = \sqrt{(\mu p)}.$$

Also

$$\dot{r} = -r^2\dot{u} = -h\frac{du}{d\theta} = \frac{\mu e}{h}\sin(\theta - \gamma).$$

But by (1) and (3)

$$\dot{r}^2 = \frac{\mu^2}{h^2}\{1 - e^2\cos^2(\theta - \gamma)\} + c.$$

Hence

$$0 = \frac{\mu^2}{h^2}(1 - e^2) + c$$

or

$$c = -\mu(1 - e^2)/p.$$

Thus if $2a$ is the transverse axis of the orbit, $c = -\mu/a$ for an ellipse, $c = 0$ for a parabola and $c = +\mu/a$ for an hyperbola. The equation of energy (2) becomes therefore

$$\left.\begin{aligned} v^2 &= 2\mu/r - \mu/a, \qquad &(e < 1) \\ v^2 &= 2\mu/r, \qquad &(e = 1) \\ v^2 &= 2\mu/r + \mu/a, \qquad &(e > 1) \end{aligned}\right\} \quad \text{.....................(4)}$$

Again, ψ being the angle which the direction of motion at (r, θ) makes with the radius vector (drawn towards the origin),

$$v\cos\psi = -\dot{r} = -\frac{\mu e}{h}\sin(\theta - \gamma)$$

$$v\sin\psi = r\dot{\theta} = hu = \frac{\mu}{h}\{1 + e\cos(\theta - \gamma)\}$$

are the components of the velocity along the radius vector (inwards) and perpendicular to it. The form of these expressions is to be noted. For they evidently represent (a) a constant velocity $V = \mu/h = \sqrt{(\mu/p)}$ perpendicular to

the radius vector, and (*b*) a constant velocity eV in a direction making an angle $\frac{1}{2}\pi+\theta-\gamma$ with the radius vector, that is, perpendicular to the transverse axis. Thus at perihelion the velocity is $V(1+e)$ and at aphelion (in the case of elliptic motion) the velocity is $V(1-e)$.

Since $h = vr \sin\psi$, the preceding equations may be written

$$\mu e \sin(\theta-\gamma) = -v^2 r \sin\psi \cos\psi$$

$$\mu e \cos(\theta-\gamma) = v^2 r \sin^2\psi - \mu$$

giving e and γ when v and ψ are given at (r, θ). Thus

$$\mu^2(e^2-1) = v^2 r (v^2 r - 2\mu) \sin^2\psi.$$

27. In finding the relations which subsist between positions in an orbit and the time it is necessary to consider separately the three kinds of conic section. The closed orbit, or ellipse, will be discussed first.

The line $\theta = \gamma$ is drawn from the pole (the Sun) in the direction of perihelion. The angle $\theta-\gamma$ is measured from this line and is called the *true anomaly*. Let it be denoted by w. Then, if t_0 is the time at perihelion,

$$t - t_0 = h^{-1}\int_\gamma r^2 d\theta$$

$$= \frac{h^3}{\mu^2}\int_0 \frac{dw}{(1+e\cos w)^2}.$$

The corresponding result in terms of the eccentric anomaly E has already been found (§ 5). It will be convenient to write down the relations between the radius vector and the true and eccentric anomalies in the forms which are most frequently required. We have

$$x = r\cos w = a(\cos E - e)$$

$$y = r\sin w = a\sqrt{(1-e^2)}\sin E.$$

Hence

$$r = \frac{a(1-e^2)}{1+e\cos w} = a(1-e\cos E) \quad\text{......................(5)}$$

$$r\cos^2\tfrac{1}{2}w = a(1-e)\cos^2\tfrac{1}{2}E$$

$$r\sin^2\tfrac{1}{2}w = a(1+e)\sin^2\tfrac{1}{2}E$$

$$\tan\tfrac{1}{2}w = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}E \quad\text{.................(6)}$$

This last equation may be regarded as the standard form of the relation between w and E. If we write $e = \sin\phi$ ($0° < \phi < 90°$), as is commonly done, then

$$\tan\tfrac{1}{2}w = \tan(45° + \tfrac{1}{2}\phi)\tan\tfrac{1}{2}E$$

$$\tan\tfrac{1}{2}E = \tan(45° - \tfrac{1}{2}\phi)\tan\tfrac{1}{2}w$$

where $\frac{1}{2}w$ and $\frac{1}{2}E$ are always in the same quadrant. Also

$$\cos w = \frac{\cos E - e}{1 - e\cos E}, \qquad \cos E = \frac{e + \cos w}{1 + e\cos w}$$

$$\sin w = \frac{\sqrt{(1-e^2)}\sin E}{1 - e\cos E}, \quad \sin E = \frac{\sqrt{(1-e^2)}\sin w}{1 + e\cos w}$$

and it readily follows that

$$dw = \frac{\sqrt{(1-e^2)}\,dE}{1 - e\cos E}, \quad dE = \frac{\sqrt{(1-e^2)}\,dw}{1 + e\cos w} \quad \text{.............(7)}$$

If now we employ (5) and (7) we obtain

$$\begin{aligned} t - t_0 &= \frac{h^3}{\mu^2}\int_0 \frac{dw}{(1 + e\cos w)^2} \\ &= \sqrt{\left(\frac{p^3}{\mu}\right)}\int_0 \frac{dE}{\sqrt{(1-e^2)}} \cdot \frac{1 - e\cos E}{1 - e^2} \\ &= \sqrt{\left(\frac{a^3}{\mu}\right)}(E - e\sin E). \end{aligned}$$

But $\mu = n^2a^3$ where n is the mean motion; the angle $n(t - t_0)$ is called the *mean anomaly* and may be denoted by M. We have therefore once more obtained Kepler's equation

$$M = n(t - t_0) = E - e\sin E \quad \text{........................(8)}$$

the angles M and E being expressed in circular measure; or if M and E are expressed in degrees, e must also be converted to the same form by the factor $180^\circ/\pi$.

28. The complete solution of the problem of elliptic motion is contained in the equations given above. No difficulty in numerical solution arises except in the case of Kepler's equation when E is to be found for given values of e and M. The general method applicable in such cases may be illustrated here. By some means an approximate solution E_0 is found. Let $E_0 + \Delta E_0$ be the exact solution, and

$$M_0 = E_0 - e\sin E_0.$$

Then

$$M = M_0 + (1 - e\cos E_0)\,\Delta E_0 + \ldots$$

when $E - e\sin E$ is expanded in a power series in ΔE_0 by Taylor's theorem. Neglecting higher powers of ΔE_0 we have

$$\Delta E_0 = (M - M_0)/(1 - e\cos E_0)$$

and hence a second approximation $E_1 = E_0 + \Delta E_0$. If this value is not sufficiently accurate the process may be repeated until a satisfactory result is obtained.

In order to obtain a good approximate solution at the outset a great variety of methods have been devised. These depend upon (*a*) the use of special tables, (*b*) an approximate formula or a series, or (*c*) a graphical method. Thus to the first order in *e*,

$$E_0 = M + e \sin M$$

and to the second order in *e*

$$\tan E_0 = \sec \phi \tan 2\chi$$

where

$$\tan \chi = \tan (45^\circ + \tfrac{1}{2}\phi) \tan \tfrac{1}{2}M$$

the verification of which may be left as an exercise.

Among graphical methods we can refer only to one, given by Newton (*Principia*, Book I, Prop. XXXI). Consider a circle of unit radius and centre C rolling on a straight line OX. Let E be the point of contact and A the point on the circumference initially coinciding with O. Let P be a point on the radius CA such that $CP = e$ and M and N the feet of the perpendiculars from P on OX and CE. Then if $E = \angle ACE = \text{arc } AE = OE$,

$$OM = OE - ME = OE - PN = E - e \sin E.$$

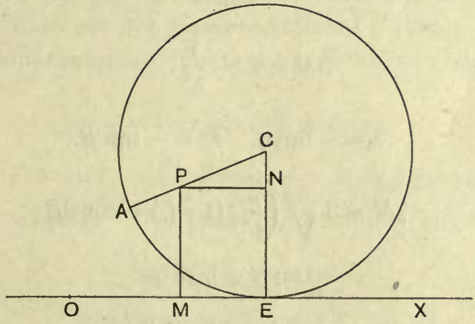


Fig. 1.

Hence if the circle is rolled (without slipping) along OX until the point P is on the ordinate PM where $OM = M$, the point of contact gives $OE = E$, which can therefore be read off when M is given. The locus of P is evidently a trochoid. It may also be noted that the ordinate

$$PM = CE - CN = 1 - e \cos E$$

which is the corresponding value of r/a or of dM/dE, and so gives the factor required for the improvement of an approximate value E_0. For references to practical applications of the above principle see *Monthly Notices, R. A. S.*, LXVII, p. 67.

29. In the case of parabolic motion

$$t - t_0 = \frac{h^3}{\mu^2}\int_0 \frac{dw}{(1+\cos w)^2}$$

$$= \sqrt{\left(\frac{p^3}{\mu}\right)}\int_0 \tfrac{1}{2}(1+\tan^2 \tfrac{1}{2}w)\, d(\tan \tfrac{1}{2}w)$$

$$= \tfrac{1}{2}\sqrt{\left(\frac{p^3}{\mu}\right)}(\tan \tfrac{1}{2}w + \tfrac{1}{3}\tan^3 \tfrac{1}{2}w)$$

and therefore a quantity M may be defined by the relation

$$M = 2\sqrt{\left(\frac{\mu}{p^3}\right)}(t - t_0) = \tan \tfrac{1}{2}w + \tfrac{1}{3}\tan^3 \tfrac{1}{2}w \quad \text{..............(9)}$$

A table, known as Barker's Table, gives M (or M multiplied by a certain numerical factor) with the argument w. An inverse table giving w with the argument M will be found in Bauschinger's *Tafeln* (No. XV). Or w may be deduced when $t - t_0$ is given thus. The equation (9) may be compared with the identity

$$\tfrac{1}{3}\left(\lambda^3 - \frac{1}{\lambda^3}\right) = \lambda - \frac{1}{\lambda} + \tfrac{1}{3}\left(\lambda - \frac{1}{\lambda}\right)^3.$$

Hence

$$\tan \tfrac{1}{2}w = \lambda - \frac{1}{\lambda}$$

if

$$3M = \lambda^3 - \frac{1}{\lambda^3}.$$

Let

$$\lambda = -\tan\gamma, \quad \lambda^3 = -\tan\beta.$$

Then

$$\tfrac{3}{2}M = 3\sqrt{\left(\frac{\mu}{p^3}\right)}(t - t_0) = \cot 2\beta$$

$$\tan\beta = \tan^3\gamma$$

and

$$\tan \tfrac{1}{2}w = 2\cot 2\gamma.$$

By these equations w can be calculated directly when t is given.

30. Hyperbolic motion along the concave branch of the curve under attraction to the focus may be treated in an analogous way to elliptic motion by using hyperbolic functions instead of circular functions of the eccentric anomaly. Thus we have

$$x = r\cos w = a(e - \cosh F)$$

$$y = r\sin w = a\sqrt{(e^2-1)}\sinh F$$

so that

$$r = \frac{a(e^2-1)}{1 + e\cos w} = a(e\cosh F - 1) \quad \text{....................(10)}$$

$$r \cos^2 \tfrac{1}{2}w = a\,(e-1)\cosh^2 \tfrac{1}{2}F$$

$$r \sin^2 \tfrac{1}{2}w = a\,(e+1)\sinh^2 \tfrac{1}{2}F$$

$$\tan \tfrac{1}{2}w = \sqrt{\left(\frac{e+1}{e-1}\right)} \tanh \tfrac{1}{2}F \quad \text{..................(11)}$$

$$\cos w = \frac{e - \cosh F}{e \cosh F - 1}, \qquad \cosh F = \frac{e + \cos w}{1 + e \cos w}$$

$$\sin w = \frac{\sqrt{(e^2-1)}\sinh F}{e \cosh F - 1}, \qquad \sinh F = \frac{\sqrt{(e^2-1)}\sin w}{1 + e\cos w}$$

$$dw = \frac{\sqrt{(e^2-1)}\,dF}{e\cosh F - 1}, \qquad dF = \frac{\sqrt{(e^2-1)}\,dw}{1 + e\cos w} \quad \text{.........(12)}$$

By employing (10) and (12) we now obtain

$$t - t_0 = \frac{h^3}{\mu^2}\int_0 \frac{dw}{(1 + e\cos w)^2}$$

$$= \sqrt{\left(\frac{p^3}{\mu}\right)} \int_0 \frac{dF}{\sqrt{(e^2-1)}} \cdot \frac{e\cosh F - 1}{e^2 - 1}$$

$$= \sqrt{\left(\frac{a^3}{\mu}\right)} (e \sinh F - F) \quad \text{........................(13)}$$

which is the analogue of Kepler's equation for this case.

Analogy suggests the use of hyperbolic functions, but full and accurate tables of these functions are not always available. Hence it is convenient to introduce f, the Gudermannian function of F, where (Log denoting natural logarithm)

$$F = \operatorname{Log} \tan (45^\circ + \tfrac{1}{2}f)$$

or

$$\sinh F = \tan f, \quad \cosh F = \sec f, \quad \tanh \tfrac{1}{2}F = \tan \tfrac{1}{2}f.$$

We may also put $e = \sec\psi$. The principal formulae (10), (11) and (13) then become

$$r = a\,(e \sec f - 1) \quad \text{...........................(14)}$$

$$\tan \tfrac{1}{2}w = \cot \tfrac{1}{2}\psi \tan \tfrac{1}{2}f \quad \text{...........................(15)}$$

and

$$\sqrt{(\mu a^{-3})}\,(t - t_0) = e \tan f - \operatorname{Log}\tan(45^\circ + \tfrac{1}{2}f) \text{............(16)}$$

The last equation may also be written

$$\sqrt{(\mu a^{-3})}\,\lambda\,(t - t_0) = \lambda e \tan f - \log\tan(45^\circ + \tfrac{1}{2}f)$$

where log denotes common logarithm and $\log\lambda = 9{\cdot}6377843$.

Comets moving in hyperbolic orbits are few in number, and in no case does the eccentricity greatly exceed unity.

31. There are certain astronomical problems which require the consideration of repulsive forces according to the law μr^{-2} which are of the same form as gravitational attraction but differ in sense. The small particles which constitute a comet's tail are apparently subject to such forces and

finely divided meteoric matter in the solar system must move under the pressure due to the Sun's radiation. Hence we shall consider the effect of replacing $+\mu$, the acceleration at unit distance, by $-\mu'$. The differential equation of the orbit becomes

$$\frac{d^2u}{d\theta^2} + u + \frac{\mu'}{h^2} = 0$$

the integral of which is

$$\begin{aligned} u &= \frac{\mu'}{h^2}\{e\cos(\theta-\gamma) - 1\} \\ &= p^{-1}(e\cos w - 1) \quad\text{.............................(17)} \end{aligned}$$

If we restrict w to such a range of values that u (or r) is positive, this equation gives only the branch of the hyperbola convex to the centre of repulsion at the focus, just as under the same restriction the equation (10) gives only the branch concave to the centre of attraction. As compared with § 26 the signs of p and e, as well as of μ, have been changed. Hence the constant c in the equation of energy becomes

$$c = -\mu'(1-e^2)/p = +\mu'/a$$

so that the equation of energy is now

$$v^2 = \mu'/a - 2\mu'/r \quad\text{..............................(18)}$$

Also, if ψ is the angle which the direction of motion at (r, θ) makes with the radius vector drawn towards the origin,

$$v\cos\psi = -\dot{r} = h\frac{du}{d\theta} = -\frac{\mu' e}{h}\sin(\theta-\gamma)$$

$$v\sin\psi = r\dot{\theta} = hu = \frac{\mu'}{h}\{e\cos(\theta-\gamma)-1\}$$

are the components of the velocity along the inward radius vector and perpendicular to it. These are evidently equivalent to (a) a constant velocity $-V' = -\mu'/h = -\sqrt{(\mu'/p)}$ perpendicular to the radius vector, the negative sign meaning that V' is drawn in the sense opposite to that in which the radius vector is rotating, and (b) a constant velocity eV' in a direction making an angle $\frac{1}{2}\pi + \theta - \gamma$ with the radius vector, that is, perpendicular to the transverse axis. Thus at perihelion the velocity is $V'(e-1)$ as compared with the velocity $V(e+1)$ at perihelion on the concave branch under an attracting force.

If the circumstances of projection are given in the form of v and ψ at the point (r, θ), we have

$$\mu' p = h^2 = v^2r^2\sin^2\psi$$

$$\mu' e\sin(\theta-\gamma) = -v^2r\sin\psi\cos\psi$$

$$\mu' e\cos(\theta-\gamma) = v^2r\sin^2\psi + \mu'$$

which determine p, e and γ in terms of given quantities. In particular

$$\mu'^2(e^2-1) = v^2r(v^2r + 2\mu')\sin^2\psi.$$

32. Expressing the coordinates in terms of hyperbolic functions we now have, since the centre is at $(ae, 0)$,

$$x = r\cos w = a(e + \cosh F)$$

$$y = r\sin w = a\sqrt{(e^2 - 1)}\sinh F.$$

Hence

$$r = \frac{a(e^2-1)}{e\cos w - 1} = a(e\cosh F + 1) \quad \text{.........................(19)}$$

$$r\cos^2\tfrac{1}{2}w = a(e+1)\cosh^2\tfrac{1}{2}F$$

$$r\sin^2\tfrac{1}{2}w = a(e-1)\sinh^2\tfrac{1}{2}F$$

$$\tan\tfrac{1}{2}w = \sqrt{\left(\frac{e-1}{e+1}\right)}\tanh\tfrac{1}{2}F \quad \text{.................................(20)}$$

$$\cos w = \frac{e+\cosh F}{e\cosh F+1}, \qquad \cosh F = \frac{e-\cos w}{e\cos w - 1}$$

$$\sin w = \frac{\sqrt{(e^2-1)}\sinh F}{e\cosh F+1}, \qquad \sinh F = \frac{\sqrt{(e^2-1)}\sin w}{e\cos w - 1}$$

$$dw = \frac{\sqrt{(e^2-1)}\,dF}{e\cosh F+1}, \qquad dF = \frac{\sqrt{(e^2-1)}\,.\,dw}{e\cos w - 1} \quad \text{......(21)}$$

It then follows that

$$t - t_0 = \int\frac{r^2}{h}d\theta = \frac{h^3}{\mu'^2}\int_0\frac{dw}{(e\cos w - 1)^2}$$

$$= \sqrt{\left(\frac{p^3}{\mu'}\right)}\int_0\frac{dF}{\sqrt{(e^2-1)}}\cdot\frac{e\cosh F+1}{e^2-1}$$

$$= \sqrt{\left(\frac{a^3}{\mu'}\right)}(e\sinh F + F) \quad \text{.................................(22)}$$

which corresponds to Kepler's equation for this case.

As in the case of an attracting force we may now put

$$\tan\tfrac{1}{2}f = \tanh\tfrac{1}{2}F, \quad \sec f = \cosh F, \quad \tan f = \sinh F$$

and $e = \sec\psi$. With these transformations the principal formulae of the solution become

$$r = a(e\sec f + 1) \quad \text{..............................(23)}$$

$$\tan\tfrac{1}{2}w = \tan\tfrac{1}{2}\psi\tan\tfrac{1}{2}f \quad \text{............................(24)}$$

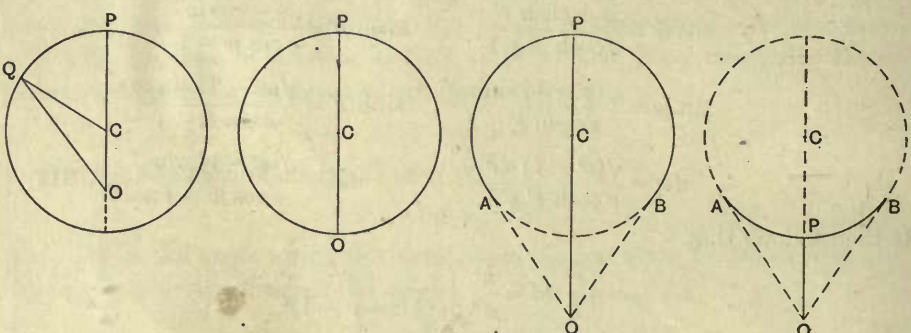
$$\sqrt{(\mu' a^{-3})}(t - t_0) = e\tan f + \operatorname{Log}\tan(45^\circ + \tfrac{1}{2}f) \quad \text{............(25)}$$

or, as the last may be written,

$$\sqrt{(\mu' a^{-3})}\,\lambda\,(t - t_0) = \lambda e\tan f + \log\tan(45^\circ + \tfrac{1}{2}f)$$

in the notation previously explained.

33. The simple and important representation of the velocity in all cases as the resultant of two vectors both constant in magnitude, and one constant in direction also, may be illustrated by considering the hodograph of the motion. This curve is clearly a circle of radius V and centre at a distance eV from the origin. The four figures given correspond with the four distinct types of motion, (*a*) elliptic, (*b*) parabolic, (*c*) hyperbolic, under attraction to the focus, and (*d*) hyperbolic, under repulsion from the focus. In all cases O is the origin, C the centre, and OP represents the velocity at perihelion. If Q is any point on the hodograph, OQ represents the velocity in the orbit at one extremity of the focal chord which is at right angles to CQ. The radius CP being V, $OC = eV$ and as the eccentricity increases O moves along the radius opposite to CP from the position C for a circular orbit to a point on the circumference for a parabolic orbit. As e increases beyond the value 1



(*a*) (*b*) Fig. 2. (*c*) (*d*)

the point O passes outside the circle. But the hodograph corresponding to hyperbolic motion is no longer a complete circle since the possible directions of motion are limited by the asymptotes. If OA, OB are the tangents from O to the circle the angles COA, COB are each equal to $\sin^{-1} e^{-1}$ and it is easily seen that OA, OB are parallel to the asymptotes of the orbit, that AOB is equal to the exterior angle between the asymptotes, and that the arc APB constitutes the whole hodograph. When the attraction is changed to a repulsion and motion takes place along the convex instead of the concave branch of the hyperbola, $OP = V'(e-1)$, and the hodograph is confined to that arc of the circle which is at all points convex to O, whereas in case (*c*) it was everywhere concave to O.

34. From the point of view of practical calculation there are points connected with orbits nearly parabolic in form which require special attention. Kepler's equation for elliptic motion may be written

$$M = E - \sin E + (1-e)\sin E.$$

When $1-e$ is small the accurate calculation of M depends on that of $E - \sin E$. But if E is small the latter expression is the difference of two nearly equal quantities and cannot be calculated directly unless each is

expressed by a disproportionate number of significant figures. Hence the need for special tables (e.g. Bauschinger's *Tafeln*, No. XL) or an approximate formula. Under the latter head may be mentioned the function

$$\tfrac{1}{6}E^3(\cos\tfrac{1}{12}E)^{14\cdot 4}$$

which is so close an approximation to $E - \sin E$ over the range of E from 0° to 70° that the logarithms of the two expressions never differ by more than 2 in the seventh place.

It is evident that in the parabola itself E is evanescent and generally in the ellipse of great eccentricity E is small at all points near the attracting focus. The method given by Gauss in the *Theoria Motus* for the treatment of Kepler's equation is a particularly instructive example of the construction and use of special tables and as at the same time it brings out clearly the relation to parabolic motion its principle will be explained here.

Kepler's equation may be written in the form

$$M = (1-e)(\alpha E + \beta \sin E) + (\beta + \alpha e)(E - \sin E)$$

if $\alpha + \beta = 1$, or

$$M = (1-e) \,.\, 2A^{\frac{1}{2}}B + (\beta + \alpha e) \,.\, \tfrac{4}{3}A^{\frac{3}{2}}B \quad \text{......................(26)}$$

if

$$A = 3\,(E - \sin E)/2\,(\alpha E + \beta \sin E)$$

and

$$B^2 = (\alpha E + \beta \sin E)^3/6\,(E - \sin E)$$

$$= (E^3 - \tfrac{1}{2}\beta \,.\, E^5 \ldots)/(E^3 - \tfrac{1}{20}E^5 \ldots)$$

which differs from unity by a quantity of the fourth order only in E if $\beta = 1/10$, $\alpha = 9/10$. With these values it is readily found that

$$A = \tfrac{1}{4}E^2 - \tfrac{1}{120}E^4 - \ldots$$

$$B = 1 + \tfrac{3}{2800}E^4 - \ldots.$$

Hence $\log B$ is a small quantity of the fourth order which is tabulated with A, itself of the second order, as argument.

We now put, in view of (26),

$$A^{\frac{1}{2}} = \sqrt{\left(\frac{5-5e}{1+9e}\right)} \tan \tfrac{1}{2}w_1$$

so that

$$M = 2\sqrt{5}\,(1-e)^{\frac{3}{2}}\,(1+9e)^{-\frac{1}{2}}\,B\,(\tan\tfrac{1}{2}w_1 + \tfrac{1}{3}\tan^3\tfrac{1}{2}w_1).$$

But

$$M = \sqrt{\left(\frac{\mu}{a^3}\right)}(t - t_0) = \sqrt{\left(\frac{\mu}{q^3}\right)}(1-e)^{\frac{3}{2}}(t - t_0)$$

where q is the perihelion distance, in the present problem a more convenient element than the mean distance a. Hence

$$\sqrt{\left(\frac{\mu}{q^3} \cdot \frac{1+9e}{20}\right)} \cdot \frac{t - t_0}{B} = \tan\tfrac{1}{2}w_1 + \tfrac{1}{3}\tan^3\tfrac{1}{2}w_1$$

the analogy of which with (9) of § 29 is evident. Here B is unknown, but the supposition that $B = 1$ will lead to a good first approximation to $\tan \frac{1}{2}w_1$ and hence to A, and a nearer value for $\log B$ can then be taken from the table. This in turn will lead to a second approximation to $\tan \frac{1}{2}w_1$, and so on until the correct value is reached. Now let

$$\tau = \tan^2 \tfrac{1}{2}E = (\tfrac{1}{2}E + \tfrac{1}{24}E^3 \ldots)^2 = \tfrac{1}{4}E^2 + \tfrac{1}{24}E^4 \ldots$$
$$= A + \tfrac{4}{5}A^2 \ldots$$

or

$$A = \tau (1 + \tfrac{4}{5}A \ldots)^{-1} = \tau (1 - \tfrac{4}{5}A + C)$$

where C is a function of the second order in A, i.e. a small quantity of the fourth order in E, which like $\log B$ can be tabulated with the argument A. Hence

$$\tan \tfrac{1}{2}w = \sqrt{\tau} \,.\, \sqrt{\left(\frac{1+e}{1-e}\right)} = \sqrt{\left(\frac{1+e}{1-e} \cdot \frac{A}{1 - \frac{4}{5}A + C}\right)}$$
$$= \tan \tfrac{1}{2}w_1 \sqrt{\left(\frac{5+5e}{1+9e}\right)} (1 - \tfrac{4}{5}A + C)^{-\frac{1}{2}}.$$

Finally, by § 27,

$$r \cos^2 \tfrac{1}{2}w = a(1-e) \cos^2 \tfrac{1}{2}E = q/(1+\tau)$$

or

$$r = \frac{1 - \frac{4}{5}A + C}{1 + \frac{1}{5}A + C} \cdot q \sec^2 \tfrac{1}{2}w$$

so that the problem of finding w and r is solved by the aid of the tables giving $\log B$ and C with the argument A without introducing E explicitly into the calculation. The method with very little change is adapted equally to hyperbolic orbits. The tables will be found in the *Theoria Motus* of Gauss, or in an equivalent form in Bauschinger's *Tafeln*, Nos. XVII and XVIII.

CHAPTER IV

EXPANSIONS IN ELLIPTIC MOTION

35. The fundamental equations of elliptic motion found in the last chapter, namely

$$M = E - e\sin E, \quad e = \sin\phi \quad \text{.......(1)}$$

$$\left.\begin{aligned} \tan\tfrac{1}{2}w &= \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}E = \tan\left(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi\right)\tan\tfrac{1}{2}E \\ &= \frac{1+\beta}{1-\beta}\tan\tfrac{1}{2}E, \quad \beta = \tan\tfrac{1}{2}\phi \end{aligned}\right\} \quad \text{.......(2)}$$

$$\frac{r}{a} = \frac{1-e^2}{1+e\cos w} = 1 - e\cos E \quad \text{.......(3)}$$

give at once the means of calculating the coordinates at any given time. But for many purposes it is necessary to express them as periodic functions in the form of series. Some of the more important forms of expansion will now be investigated.

But certain changes in these equations are sometimes useful. Let

$$\iota w = \log x, \quad \iota E = \log y, \quad \iota M = \log z, \quad \iota^2 = -1.$$

Then from (2)

$$\frac{x-1}{x+1} = \frac{1+\beta}{1-\beta}\cdot\frac{y-1}{y+1}$$

$$x = \frac{y-\beta}{1-\beta y}, \quad y = \frac{x+\beta}{1+\beta x}.$$

Also by (1)

$$\log z = \log y - \tfrac{1}{2}e\left(y - y^{-1}\right)$$

$$z = y\exp.\left[-\tfrac{1}{2}e\left(y-y^{-1}\right)\right] \quad \text{.......(4)}$$

$$= \frac{x+\beta}{1+\beta x}\exp.\left[\frac{-\beta}{1+\beta^2}\cdot\frac{(x^2-1)(1-\beta^2)}{(x+\beta)(1+\beta x)}\right]$$

$$= x\left(1+\beta x^{-1}\right)\left(1+\beta x\right)^{-1}\exp.\left[\beta\cos\phi\left\{(\beta+x)^{-1} - \left(\beta+x^{-1}\right)^{-1}\right\}\right] \text{...(5)}$$

The equation (3) gives

$$\left.\begin{aligned}\frac{r}{a} &= 1 - \frac{\beta}{1+\beta^2}(y+y^{-1}) = \frac{1}{1+\beta^2}(1-\beta y)(1-\beta y^{-1}) \\ &= \frac{1}{1+\beta^2}\cdot\frac{1-\beta^2}{1+\beta x}\cdot\frac{x(1-\beta^2)}{x+\beta} = \frac{(1-\beta^2)^2}{1+\beta^2}\cdot(1+\beta x)^{-1}(1+\beta x^{-1})^{-1}\end{aligned}\right\}\quad\text{...(6)}$$

It is evident that some expansions will be made more simply in terms of β than of e. Hence it will be useful to have the development of any positive power of β in terms of e. Now

$$\beta + \beta^{-1} = \tan\tfrac{1}{2}\phi + \cot\tfrac{1}{2}\phi = 2\operatorname{cosec}\phi = 2e^{-1}$$

or

$$\beta = 0 + \tfrac{1}{2}e(1+\beta^2).$$

Hence by Lagrange's theorem

$$\begin{aligned}\beta^m &= \sum_q \frac{(\frac{1}{2}e)^q}{q\,!}\left[\frac{d^{q-1}}{dx^{q-1}}\{mx^{m-1}(1+x^2)^q\}\right]_{x=0} \\ &= m\sum_q \frac{(\frac{1}{2}e)^q}{q\,!}\left[\frac{d^{q-1}}{dx^{q-1}}\sum_p \binom{q}{p} x^{2p+m-1}\right]_{x=0} \\ &= m\sum_p \frac{(\frac{1}{2}e)^{2p+m}}{(2p+m)\,!}\left[\frac{d^{2p+m-1}}{dx^{2p+m-1}}\binom{2p+m}{p} x^{2p+m-1}\right]_{x=0}\end{aligned}$$

for the only terms which survive arise when $q = 2p+m$. Hence

$$\begin{aligned}\beta^m &= m\sum_{p=0} (\tfrac{1}{2}e)^{2p+m}\frac{(2p+m-1)\,!}{p\,!\,(p+m)\,!} \\ &= (\tfrac{1}{2}e)^m\left\{1 + \frac{m}{4}\cdot e^2 + \frac{m}{4^2}\cdot\frac{m+3}{2\,!}e^4 + \frac{m}{4^3}\cdot\frac{(m+4)(m+5)}{3\,!}e^6 + \dots\right\}\end{aligned}\quad\text{...(7)}$$

and it is readily seen that this series is absolutely convergent.

36. Since

$$x = (y-\beta)(1-\beta y)^{-1}$$

it follows that

$$\begin{aligned}\log x &= \log y + \log(1-\beta y^{-1}) - \log(1-\beta y) \\ &= \log y + \beta(y-y^{-1}) + \tfrac{1}{2}\beta^2(y^2-y^{-2}) + \dots.\end{aligned}$$

Hence

$$w = E + 2(\beta\sin E + \tfrac{1}{2}\beta^2\sin 2E + \tfrac{1}{3}\beta^3\sin 3E + \dots)\quad\text{.........(8)}$$

But x and y can be interchanged if the sign of β is changed at the same time. Therefore

$$E = w - 2(\beta\sin w - \tfrac{1}{2}\beta^2\sin 2w + \tfrac{1}{3}\beta^3\sin 3w - \dots).$$

It is also easy to express M in terms of w. For, by (5),

$$\begin{aligned}\log z &= \log x + \log(1+\beta x^{-1}) - \log(1+\beta x) + \beta\cos\phi\{(x+\beta)^{-1} - (x^{-1}+\beta)^{-1}\} \\ &= \log x - \beta(x-x^{-1}) + \tfrac{1}{2}\beta^2(x^2-x^{-2}) - \tfrac{1}{3}\beta^3(x^3-x^{-3}) + \dots \\ &\qquad + \beta\cos\phi\{-(x-x^{-1}) + \beta(x^2-x^{-2}) - \beta^2(x^3-x^{-3}) + \dots\} \\ &= \log x - \beta(1+\cos\phi)(x-x^{-1}) + \beta^2(\tfrac{1}{2}+\cos\phi)(x^2-x^{-2}) - \dots\end{aligned}$$

and therefore

$$M = w - 2\{\beta(1+\cos\phi)\sin w - \beta^2(\tfrac{1}{2}+\cos\phi)\sin 2w + \beta^3(\tfrac{1}{3}+\cos\phi)\sin 3w - \ldots\}.$$

By this expansion the *equation of the centre*, $w - M$, is expressed as a series in terms of the true anomaly.

37. We have now to consider the expansions in terms of M, which are of the greatest importance because they are required in order to express the coordinates as periodic functions of the time. And first we take the case of r^{-1}. Now

$$\frac{a}{r} = (1 - e\cos E)^{-1} = \frac{dE}{dM}.$$

This is an even periodic function of E and consequently of M. Hence

$$\frac{a}{r} = \frac{1}{\pi}\int_0^\pi (1-e\cos E)^{-1}\,dM + \Sigma\frac{2}{\pi}\cos pM\int_0^\pi (1-e\cos E)^{-1}\cos pM\,dM$$

$$= \frac{1}{\pi}\int_0^\pi dE + \frac{2}{\pi}\Sigma\cos pM\int_0^\pi \cos(pE - pe\sin E)\,dE$$

$$= 1 + 2\sum_{p=1}^{\infty} J_p(pe)\cos pM \quad\ldots\ldots(9)$$

where

$$J_p(pe) = \frac{1}{\pi}\int_0^\pi \cos(pE - pe\sin E)\,dE.$$

$J_p(pe)$ is called the *Bessel's coefficient* of order p and argument pe. We shall briefly study the properties of these coefficients so far as they are required for our immediate purpose.

Let

$$F(t) = \exp.\{\tfrac{1}{2}x(t - t^{-1})\} = \sum_{-\infty}^{+\infty} a_p t^p.$$

For t write $\exp.(-\iota\psi)$. Then

$$\exp.(-\iota x\sin\psi) = \sum_{-\infty}^{+\infty} a_p \exp.(-\iota p\psi).$$

This is a Fourier expansion, showing that

$$a_p = \frac{1}{2\pi}\int_0^{2\pi} \exp.\iota(p\psi - x\sin\psi)\,d\psi$$

and combining the parts of the integral which are due to ψ and $2\pi - \psi$ we have

$$a_p = \frac{1}{\pi}\int_0^\pi \cos(p\psi - x\sin\psi)\,d\psi$$

$$= J_p(x) \quad\ldots\ldots(10)$$

Thus the coefficients in the expansion of $F(t)$ are precisely the coefficients which we have to study. Now

$$\begin{aligned} F(t) &= \exp. (\tfrac{1}{2}xt) \exp. (-\tfrac{1}{2}xt^{-1}) \\ &= \Sigma (\tfrac{1}{2}x)^{\alpha} . \frac{t^{\alpha}}{\alpha!} . \Sigma (-1)^{\beta} . (\tfrac{1}{2}x)^{\beta} . \frac{t^{-\beta}}{\beta!} . \\ &= \Sigma\Sigma (-1)^{\beta} (\tfrac{1}{2}x)^{\alpha+\beta} \frac{t^{\alpha-\beta}}{\alpha!\,\beta!} . \end{aligned}$$

Hence $J_p(x)$ is the coefficient of those terms for which $\alpha = \beta + p$, or

$$J_p(x) = \Sigma \frac{(-1)^{\beta}}{\beta!\,(\beta+p)!} (\tfrac{1}{2}x)^{p+2\beta} .$$

If p is positive, β takes the values 0, 1, 2,... and the expansion becomes

$$J_p(x) = \frac{x^p}{2^p . p!} \left\{ 1 - \frac{x^2}{2 . (2p+2)} + \frac{x^4}{2 . 4 . (2p+2)(2p+4)} - \dots \right\} \dots (11)$$

If p is negative, β takes the values $-p, -p+1, \dots$, because α cannot be negative.

38. The effect of changing the signs of x and t is to leave $F(t)$ unaltered. Hence

$$J_p(x) = (-1)^p J_p(-x) \quad \dots\dots\dots\dots\dots\dots\dots\dots (12)$$

Similarly $F(t)$ is unchanged if $-t^{-1}$ is substituted for t. Hence

$$J_p(x) = (-1)^p J_{-p}(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots (13)$$

Again, the result of differentiating $F(t)$ with respect to t, gives

$$\tfrac{1}{2}x(1+t^{-2}) \Sigma J_p(x)\, t^p = \Sigma p\, J_p(x)\, t^{p-1} .$$

Equating the coefficients of t^{p-1} we have

$$\tfrac{1}{2}x \{J_{p-1}(x) + J_{p+1}(x)\} = p J_p(x) \quad \dots\dots\dots\dots\dots\dots (14)$$

On the other hand, if we differentiate $F(t)$ with respect to x, we have

$$\tfrac{1}{2}(t - t^{-1}) \Sigma J_p(x)\, t^p = \Sigma J_p'(x)\, t^p$$

or, equating the coefficients of t^p,

$$\tfrac{1}{2} \{J_{p-1}(x) - J_{p+1}(x)\} = J_p'(x) \quad \dots\dots\dots\dots\dots\dots (15)$$

These simple recurrence formulae show that, with any given argument, Bessel's coefficients of any order, and their derivatives, can be expressed as linear functions of the coefficients of any two particular orders, or of any one coefficient and its derivative, e.g. $J_1(x)$ and $J_1'(x)$. In particular,

$$\begin{aligned} J_p''(x) &= \tfrac{1}{2} \{J'_{p-1}(x) - J'_{p+1}(x)\} \\ &= \tfrac{1}{4} \{J_{p-2}(x) - 2J_p(x) + J_{p+2}(x)\} \\ &= -J_p(x) + \frac{1}{2x} \{(p-1) J_{p-1}(x) + (p+1) J_{p+1}(x)\} \\ &= -J_p(x) + \frac{p^2}{x^2} J_p(x) - \frac{1}{x} J_p'(x) \end{aligned}$$

or

$$J_p''(x) + \frac{1}{x} J_p'(x) + \left(1 - \frac{p^2}{x^2}\right) J_p(x) = 0.$$

This shows that $J_p(x)$ is a particular solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{p^2}{x^2}\right) y = 0 \quad \text{.....................(16)}$$

The general theory of Bessel's functions, defined as solutions of this differential equation, is not required for our purpose. We need only the solutions of the first kind, with integral values of p, and the definition given above is sufficient.

39. The desired expansions in M can now be resumed. We take $\sin mE$ which is an odd function of E and M. Therefore

$$\begin{aligned}
\sin mE &= \frac{2}{\pi} \Sigma \sin pM \int_0^\pi \sin mE \sin pM \, dM \\
&= -\frac{2}{\pi} \Sigma \sin pM \int_0^\pi \frac{1}{p} \sin mE \,.\, d\{\cos(pE - pe \sin E)\} \\
&= \frac{2}{\pi} \Sigma \sin pM \int_0^\pi \frac{m}{p} \cos mE \cos(pE - pe \sin E) \, dE
\end{aligned}$$

(by integration by parts, the integrated part vanishing at the limits)

$$\begin{aligned}
&= \frac{1}{\pi} \Sigma \sin pM \int_0^\pi \frac{m}{p} \{\cos(\overline{p - m}E - pe \sin E) \\
&\qquad\qquad + \cos(\overline{p + m}E - pe \sin E)\} \, dE \\
&= m \Sigma \frac{\sin pM}{p} \{J_{p-m}(pe) + J_{p+m}(pe)\} \quad \text{...............(17)}
\end{aligned}$$

In particular, when $m = 1$, by (14)

$$\sin E = \frac{2}{e} \Sigma \frac{\sin pM}{p} \,.\, J_p(pe) \quad \text{.....................(18)}$$

and therefore

$$E = M + 2 \Sigma \frac{\sin pM}{p} \,.\, J_p(pe) \quad \text{...............(19)}$$

Similarly, since $\cos mE$ is an even function of E and M,

$$\begin{aligned}
\cos mE &= a_0 + \frac{2}{\pi} \Sigma \cos pM \int_0^\pi \cos mE \cos pM \, dM \\
&= a_0 + \frac{2}{\pi} \Sigma \cos pM \int_0^\pi \frac{1}{p} \cos mE \,.\, d\{\sin(pE - pe \sin E)\} \\
&= a_0 + \frac{2}{\pi} \Sigma \cos pM \int_0^\pi \frac{m}{p} \sin mE \sin(pE - pe \sin E) \, dE
\end{aligned}$$

(integrating by parts as before)

$$= a_0 + \frac{1}{\pi} \Sigma \cos pM \int_0^\pi \frac{m}{p} \{\cos(\overline{p - m}E - pe\sin E)$$
$$- \cos(\overline{p + m}E - pe\sin E)\}\, dE$$

$$= a_0 + m\Sigma \frac{\cos pM}{p} \{J_{p-m}(pe) - J_{p+m}(pe)\} \quad \text{.................(20)}$$

The constant term has not been determined. It is

$$a_0 = \frac{1}{\pi}\int_0^\pi \cos mE\, dM$$

$$= \frac{1}{\pi}\int_0^\pi \cos mE\,(1 - e\cos E)\, dE$$

$$= \frac{1}{\pi}\int_0^\pi \{\cos mE - \tfrac{1}{2}e\cos(m+1)E - \tfrac{1}{2}e\cos(m-1)E\}\, dE$$

and thus

$$a_0 = 1 \text{ if } m = 0$$
$$= -\tfrac{1}{2}e \text{ if } m = 1$$
$$= 0 \text{ if } m > 1.$$

The particular case of $m = 1$ is simplified by (15), so that

$$\cos E = -\tfrac{1}{2}e + 2\Sigma \frac{\cos pM}{p} J_p'(pe) \quad \text{..................(21)}$$

40. From the last expansion it follows that

$$\frac{r}{a} = 1 - e\cos E = 1 + \tfrac{1}{2}e^2 - 2e\Sigma \frac{\cos pM}{p} J_p'(pe) \quad \text{.........(22)}$$

Any positive power of r can be expanded by means of (20). For example

$$\frac{r^2}{a^2} = (1 - e\cos E)^2$$

$$= 1 + \tfrac{1}{2}e^2 - 2e\cos E + \tfrac{1}{2}e^2\cos 2E$$

$$= 1 + \tfrac{1}{2}e^2 + e^2 - 4e\Sigma \frac{\cos pM}{p} J_p'(pe) + e^2\Sigma \frac{\cos pM}{p}\{J_{p-2}(pe) - J_{p+2}(pe)\}.$$

Now, by (14) and (15),

$$J_{p-2}(pe) - J_{p+2}(pe) = \frac{2(p-1)}{pe} J_{p-1}(pe) - \frac{2(p+1)}{pe} J_{p+1}(pe)$$

$$= \frac{4}{e} J_p'(pe) - \frac{4}{pe^2} J_p(pe).$$

Hence

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - 4\Sigma \frac{\cos pM}{p^2} J_p(pe) \quad \text{..................(23)}$$

The expansions of the rectangular coordinates can be written down at once by means of (18) and (21). Thus, if x, y have this meaning and not as in § 35,

$$x = a\cos E - ae$$

$$= a\left\{-\tfrac{3}{2}e + 2\Sigma\frac{\cos pM}{p}J_p'(pe)\right\} \quad\text{.................(24)}$$

and

$$y = \sqrt{(1-e^2)}\,a\sin E$$

$$= 2a\cot\phi\,\Sigma\frac{\sin pM}{p}J_p(pe) \quad\text{........................(25)}$$

Other important expansions can be derived from those already obtained by differentiation or integration. For instance, the equations of motion give directly

$$\frac{d^2x}{dM^2} + \frac{a^3x}{r^3} = 0$$

$$\frac{d^2y}{dM^2} + \frac{a^3y}{r^3} = 0$$

whence

$$\frac{x}{r^3} = \frac{2}{a^2}\Sigma\,pJ_p'(pe)\cos pM \quad\text{.........................(26)}$$

$$\frac{y}{r^3} = \frac{2}{a^2}\cot\phi\,\Sigma\,pJ_p(pe)\sin pM \quad\text{..................(27)}$$

41. The expansion of functions of the true anomaly in terms of the mean anomaly is in general more difficult. But $\sin w$ and $\cos w$ are readily found. For (§ 27)

$$\sin w = \frac{\sqrt{(1-e^2)}\sin E}{1 - e\cos E}$$

$$= \cot\phi\frac{d}{dE}(1 - e\cos E)\frac{dE}{dM}$$

$$= \cot\phi\frac{d}{dM}\left(\frac{r}{a}\right)$$

$$= 2\cos\phi\,\Sigma\,J_p'(pe)\sin pM \quad\text{.....................(28)}$$

by (22). And

$$\cos w = \frac{\cos E - e}{1 - e\cos E}$$

$$= -e^{-1} + \frac{1-e^2}{e}\cdot\frac{a}{r}$$

$$= -e + \frac{2(1-e^2)}{e}\Sigma\,J_p(pe)\cos pM \quad\text{..................(29)}$$

by (9).

Hence also for the equation of the centre,

$$\sin(w-M) = e\sin M - \frac{1-e^2}{e}\Sigma J_p(pe)\{\sin(p+1)M - \sin(p-1)M\}$$
$$+\sqrt{(1-e^2)}\,\Sigma J_p'(pe)\{\sin(p+1)M + \sin(p-1)M\}$$
$$= \left\{e + \frac{1-e^2}{e}J_2(2e) + \sqrt{(1-e^2)}\,J_2'(2e)\right\}\sin M + \sum_{p=2}^{\infty} a_p \sin pM \ldots (30)$$

where

$$a_p = -\frac{1-e^2}{e}\{J_{p-1}(\overline{p-1}\,.\,e) - J_{p+1}(\overline{p+1}\,.\,e)\}$$
$$+\sqrt{(1-e^2)}\,\{J'_{p-1}(\overline{p-1}\,.\,e) + J'_{p+1}(\overline{p+1}\,.\,e)\}.$$

This expansion for the equation of the centre in terms of the mean anomaly is important, although the coefficients are rather complicated. Hence, as far as e^3,

$$\sin(w-M) = e(2-\tfrac{5}{4}e^2)\sin M + \tfrac{5}{4}e^2\sin 2M + \tfrac{17}{12}e^3\sin 3M$$
$$w - M = e(2-\tfrac{1}{4}e^2)\sin M + \tfrac{5}{4}e^2\sin 2M + \tfrac{13}{12}e^3\sin 3M$$

as can easily be verified.

***42.** For some purposes Laurent series in the exponentials x, y, z of § 35 are more convenient than Fourier series in w, E, M. Clearly

$$x^{-1}dx = \iota\,dw, \quad y^{-1}dy = \iota\,dE, \quad z^{-1}dz = \iota\,dM.$$

Let

$$S = a_0 + \Sigma(a_p\cos p\theta + b_p\sin p\theta)$$
$$= a_0 + \Sigma\{\tfrac{1}{2}(a_p - \iota b_p)\tau^p + \tfrac{1}{2}(a_p + \iota b_p)\tau^{-p}\}$$

where $\log\tau = \iota\theta$. By Fourier's theorem

$$\pi a_p = \int_0^{2\pi} S\cos p\theta\,d\theta, \quad \pi b_p = \int_0^{2\pi} S\sin p\theta\,d\theta$$

$$\pi(a_p - \iota b_p) = \int_0^{2\pi} S\tau^{-p}\,d\theta, \quad \pi(a_p + \iota b_p) = \int_0^{2\pi} S\tau^p\,d\theta.$$

Hence

$$S = \sum_{-\infty}^{\infty} A_p\tau^p$$

where

$$2\pi A_p = \int_0^{2\pi} S\tau^{-p}\,d\theta.$$

This well-known form, intermediate between Fourier's and Laurent's, is general and includes the case $p=0$. It has been used already in § 37.

Formulae have been found which make it possible to pass from any Fourier's expansion in E to one in M. The general result may be expressed in a slightly different way. For, since y has the same period as z,

$$y^p = \Sigma A_m z^m$$

* The reading of §§ 42—46 can quite conveniently be deferred till after Chapter XIII.

where

$$2\pi A_m = \int_0^{2\pi} y^p z^{-m}\, dM = \iota m^{-1} \int y^p d\,(z^{-m})$$

$$= [-\iota m^{-1} y^p z^{-m}] - \iota p m^{-1} \int y^{p-1} z^{-m}\, dy$$

$$= p m^{-1} \int_0^{2\pi} y^p z^{-m}\, dE$$

$$= p m^{-1} \int_0^{2\pi} \exp. \{\iota p E - \iota m\,(E - e \sin E)\}\, dE$$

$$= 2\pi\, p m^{-1} J_{m-p}\,(me)$$

$(m \neq 0)$. But when $m = 0$,

$$2\pi A_0 = \int_0^{2\pi} y^p\, dM = \int_0^{2\pi} y^p\,(1 - e \cos E)\, dE$$

$$= \int_0^{2\pi} (y^p - \tfrac{1}{2} e y^{p+1} - \tfrac{1}{2} e y^{p-1})\, dE$$

$$= 2\pi\,(p = 0); \quad -\pi e\,(p = \pm 1); \quad 0\,(p^2 > 1).$$

Hence generally, for any function of y,

$$S = \Sigma\, B_p y^p = \sum_p \sum_{m=\pm 1}^{\pm\infty} B_p A_m z^m + \sum_p B_p A_0$$

$$= B_0 - \tfrac{1}{2} e\,(B_1 + B_{-1}) + \sum_{m=\pm 1}^{\pm\infty} \sum_p p m^{-1} B_p J_{m-p}\,(me)\, z^m.$$

43. There is another form of calculation, due to Cauchy, in which Bessel's coefficients do not appear explicitly. Let S be any periodic function, such that

$$S = \Sigma\, A_p z^p.$$

Here, by (4),

$$2\pi A_p = \int_0^{2\pi} S z^{-p}\, dM$$

$$= \int_0^{2\pi} S y^{-p} \exp. [\tfrac{1}{2} p e\,(y - y^{-1})]\,(1 - e \cos E)\, dE$$

$$= \int_0^{2\pi} S y^{-p} \{1 - \tfrac{1}{2} e\,(y + y^{-1})\} \exp. [\tfrac{1}{2} p e\,(y - y^{-1})]\, dE$$

$$= \int_0^{2\pi} U y^{-p}\, dE$$

where

$$U = S\,\{1 - \tfrac{1}{2} e\,(y + y^{-1})\} \exp. [\tfrac{1}{2} p e\,(y - y^{-1})] \quad \text{.................}(31)$$

$$= \Sigma\, B_p y^p$$

the coefficient B_p of U expanded in powers of $y^{\pm 1}$ being thus identical with the coefficient A_p of S expanded in powers of $z^{\pm 1}$.

Again,

$$2\pi A_p = -\iota \int_0^{2\pi} S z^{-p-1} \frac{dz}{dM}\, dM = \iota p^{-1} \int_0^{2\pi} S \frac{dz^{-p}}{dM}\, dM$$

$$= -\iota p^{-1} \int_0^{2\pi} z^{-p} \frac{dS}{dM}\, dM = -\iota p^{-1} \int_0^{2\pi} z^{-p} \frac{dS}{dE}\, dE$$

$$= p^{-1} \int_0^{2\pi} z^{-p} y \frac{dS}{dy}\, dE$$

$$= \int_0^{2\pi} \frac{1}{p} y^{-p+1} \frac{dS}{dy} \exp.\, [\tfrac{1}{2} pe (y - y^{-1})]\, dE$$

$$= \int_0^{2\pi} V y^{-p+1}\, dE$$

where

$$V = \frac{1}{p} \frac{dS}{dy} \exp.\, [\tfrac{1}{2} pe (y - y^{-1})] \quad \text{......(32)}$$

$$= \Sigma B_p' y^p$$

the coefficient B'_{p-1} of V expanded in powers of $y^{\pm 1}$ being thus identical with the coefficient A_p of S expanded in powers of $z^{\pm 1}$. The form (32) becomes illusory when $p = 0$.

Now the exponential function occurring in (31), (32) can be expanded in a series with Bessel's coefficients having the argument pe. That returns to the methods already considered. But another process is possible and has advantages if S is of suitable form. This consists in developing first in powers of $y - y^{-1}$. Let

$$(t + t^{-1})^j (t - t^{-1})^q = \sum_{p=-\infty}^{\infty} N_{-p, j, q}\, t^p$$

where j and q are integers (not negative). The numerical coefficients N are called *Cauchy's numbers* and it is evident that a knowledge of them will be required in this method. By comparing coefficients of t^p in the identity

$$(t + t^{-1})^{j+1} (t - t^{-1})^q = t^{-1} (t + t^{-1})^j (t - t^{-1})^q + t (t + t^{-1})^j (t - t^{-1})^q$$

it is evident that

$$N_{-p, j+1, q} = N_{-p-1, j, q} + N_{-p+1, j, q}.$$

From a double-entry table giving $N_{-p, 0, q}$ with the arguments p, q, therefore, similar tables giving $N_{-p, 1, q}$, $N_{-p, 2, q}$, ... can be readily constructed. The effect of interchanging t and t^{-1} shows that

$$N_{-p, j, q} = (-1)^q N_{p, j, q}.$$

The expansion is either even or odd and the highest term is t^{j+q}. Hence $j + q - p$ is a positive even integer, and if $p = j + q$, $N = 1$.

It is now only necessary to consider the construction of the table for $N_{-p,0,q}$ when p is positive. But this is indicated by

$$(t - t^{-1})^q = \Sigma N_{-p,0,q} t^p = \Sigma \frac{q!}{r!(q-r)!} t^r (-t^{-1})^{q-r}$$

whence $p = 2r - q$, and

$$N_{-p,0,q} = (-1)^{\frac{1}{2}(q-p)} \frac{q!}{[\frac{1}{2}(q+p)]! [\frac{1}{2}(q-p)]!}.$$

The tabulation of Cauchy's numbers, which are all positive or negative integers, is therefore an extremely simple matter.

44. To consider an example, let

$$S = \left(\frac{r}{a} - 1\right)^m = (-e \cos E)^m = (-\tfrac{1}{2}e)^m (y + y^{-1})^m.$$

Then

$$\begin{aligned} U &= \{(-\tfrac{1}{2}e)^m (y + y^{-1})^m + (-\tfrac{1}{2}e)^{m+1} (y + y^{-1})^{m+1}\} \exp. [\tfrac{1}{2}pe(y - y^{-1})] \\ &= \{(-\tfrac{1}{2}e)^m (y + y^{-1})^m + (-\tfrac{1}{2}e)^{m+1} (y + y^{-1})^{m+1}\} \sum_q (\tfrac{1}{2}pe)^q (y - y^{-1})^q / q! \\ &= (-\tfrac{1}{2}e)^m (y + y^{-1})^m \sum_q (\tfrac{1}{2}pe)^q (y - y^{-1})^q / q! \\ &\qquad + (-\tfrac{1}{2}e)^{m+1} (y + y^{-1})^{m+1} \sum_q (\tfrac{1}{2}pe)^{q-1} (y - y^{-1})^{q-1} / (q-1)! \end{aligned}$$

and

$$B_p = (-\tfrac{1}{2}e)^m \sum_q \frac{(\frac{1}{2}pe)^q}{q!} \left[N_{-p,m,q} - \frac{q}{p} N_{-p,m+1,q-1}\right]$$

is the coefficient of y^p in U, and therefore of z^p in S.

When $p = 0$ the exponential function disappears and the constant term is given by

$$U = (-\tfrac{1}{2}e)^m (y + y^{-1})^m + (-\tfrac{1}{2}e)^{m+1} (y + y^{-1})^{m+1}$$

and is therefore the first or the second of the forms

$$(\tfrac{1}{2}e)^m \, m! \, [(\tfrac{1}{2}m)!]^{-2}, \quad (\tfrac{1}{2}e)^{m+1} (m+1)! \, \{[\tfrac{1}{2}(m+1)]!\}^{-2}$$

according as m is even or odd.

On the other hand,

$$\frac{dS}{dy} = m(-\tfrac{1}{2}e)^m y^{-1} (y - y^{-1}) (y + y^{-1})^{m-1}$$

and therefore

$$V = \frac{m}{p} (-\tfrac{1}{2}e)^m y^{-1} (y + y^{-1})^{m-1} \sum_q \frac{(\frac{1}{2}pe)^q}{q!} (y - y^{-1})^{q+1}.$$

Hence

$$B'_{p-1} = \frac{m}{p} (-\tfrac{1}{2}e)^m \sum_q \frac{(\frac{1}{2}pe)^q}{q!} N_{-p,m-1,q+1}$$

is the coefficient of y^{p-1} in V and therefore also the coefficient of z^p in S. Comparison with the previous result shows that

$$mN_{-p,\,m-1,\,q+1} = pN_{-p,\,m,\,q} - qN_{-p,\,m+1,\,q-1}$$

is an identity. From this the recurrence formula

$$(m-p+q+2)\,N_{-p+2,\,m,\,q} - 2\,(m-q)\,N_{-p,\,m,\,q} + (m+p+q+2)\,N_{-p-2,\,m,\,q}$$

can be easily deduced.

45. The development in terms of M or z of the functions

$$\left(\frac{r}{a}\right)^n \begin{matrix}\sin\\ \cos\end{matrix}\, mw, \quad \left(\frac{r}{a}\right)^n x^m$$

is of special importance. Here n is any positive or negative integer, and if m is also a positive or negative integer it is only necessary to consider the second form. This involves *Hansen's coefficients* $X_i^{n,\,m}$, where

$$\left(\frac{r}{a}\right)^n x^m = \Sigma\, X_i^{n,\,m} z^i, \qquad 2\pi X_i^{n,\,m} = \int_0^{2\pi} \left(\frac{r}{a}\right)^n x^m z^{-i}\, dM.$$

Now

$$dM = \frac{r}{a}\, dE = \left(\frac{r}{a}\right)^2 \sec\phi\, dw = \frac{1+\beta^2}{1-\beta^2}\left(\frac{r}{a}\right)^2 dw$$

of which the last form follows from the areal property of elliptic motion,

$$r^2 dw = h\,dt = n^{-1} h\, dM = ab\,.\,dM = a^2 \cos\phi\, dM.$$

Also

$$x = y\,(1-\beta y^{-1})(1-\beta y)^{-1}$$

and therefore $X_i^{n,\,m}$ can be expressed by a definite integral involving y and E, or by one involving x and w, by means of (4), (5), (6), thus

$$2\pi X_i^{n,\,m} = \int_0^{2\pi} (1+\beta^2)^{-n-1} y^{m-i} (1-\beta y)^{n+1-m} (1-\beta y^{-1})^{n+1+m} \exp.\, [\tfrac{1}{2} ie\,(y - y^{-1})]\, dE$$

and

$$2\pi X_i^{n,\,m} = \int_0^{2\pi} (1-\beta^2)^{2n+3} (1+\beta^2)^{-n-1} x^{m-i} (1+\beta x)^{-n-2+i} (1+\beta x^{-1})^{-n-2-i} \exp.\, [i\beta \cos\phi\, \{(\beta + x^{-1})^{-1} - (\beta + x)^{-1}\}]\, dw.$$

The first of these forms shows that $(1+\beta^2)^{n+1} X_i^{n,\,m}$ is the coefficient of y^{i-m} in the expanded product $Y_1 Y_2$, where

$$Y_1 = (1-\beta y)^{n+1-m} \exp.\, (\tfrac{1}{2} iey)$$
$$Y_2 = (1-\beta y^{-1})^{n+1+m} \exp.\, (-\tfrac{1}{2} iey^{-1}).$$

Similarly the second form shows that $(1+\beta^2)^{n+1}(1-\beta^2)^{-2n-3} X_i^{n,\,m}$ is the coefficient of x^{i-m} in the expanded product $X_1 X_2$, where

$$X_1 = (1+\beta x)^{-n-2+i} \exp.\, [i \cos\phi\,.\,\beta x\,(1+\beta x)^{-1}]$$
$$X_2 = (1+\beta x^{-1})^{-n-2-i} \exp.\, [-i\cos\phi\,.\,\beta x^{-1}(1+\beta x^{-1})^{-1}].$$

The deduction of Hansen's formulae in this way is not difficult, and has been given by Tisserand (*Méc. Cél.*, I, ch. XV).

An obvious method consists in expanding the exponential function occurring in the first of the two integral forms in a series with Bessel's coefficients. Thus

$$2\pi X_i^{n,m} = (1+\beta^2)^{-n-1} \sum_p J_p(ie) \int_0^{2\pi} y^{p+m-i} (1-\beta y)^{n+1-m} (1-\beta y^{-1})^{n+1+m}\, dE$$

$$= 2\pi (1+\beta^2)^{-n-1} \sum_p J_p(ie) X_{i,p}^{n,m}$$

where $X_{i,p}^{n,m}$ is clearly the coefficient of y^{i-p-m} in the expansion of

$$Y_m^n(\beta) = (1-\beta y)^{n+1-m} (1-\beta y^{-1})^{n+1+m}$$

and therefore equally the coefficient of y^{-i+p+m} in the expansion of

$$Y_{-m}^n(\beta) = (1-\beta y^{-1})^{n+1-m} (1-\beta y)^{n+1+m}.$$

Now

$$(1-\beta y)^i (1-\beta y^{-1})^j = \Sigma (-\beta)^{h+k} y^{h-k} . \frac{i \dots (i-h+1)}{h!} . \frac{j \dots (j-k+1)}{k!}$$

$$= \sum_k (-1)^p \beta^{p+2k} y^p \frac{i \dots (i-p-k+1)}{(p+k)!} . \frac{j \dots (j-k+1)}{k!}$$

where $h = p+k$, and if j is positive the coefficient of y^p is

$$(-\beta)^p . \frac{i \dots (i-p+1)}{p!} \sum_k \frac{(i-p) \dots (i-p-k+1)}{(p+1) \dots (p+k)} . \frac{j \dots (j-k+1)}{k!} \beta^{2k}$$

$$= (-\beta)^p \binom{i}{p} F(p-i, -j, p+1, \beta^2)$$

in the ordinary notation for a hypergeometric series. Hence there are two possible forms for $X_{i,p}^{n,m}$:

$$(-\beta)^{i-p-m} \binom{n+1-m}{i-p-m} F(i-p-n-1, -m-n-1, i-p-m+1, \beta^2)$$

$$(-\beta)^{-i+p+m} \binom{n+1+m}{-i+p+m} F(-i+p-n-1, m-n-1, -i+p+m+1, \beta^2)$$

of which the first is available if $i-p-m > 0$ and the second if $i-p-m < 0$, for then the third argument of the series is positive and the binomial coefficient has a meaning. If $i-p = m$ both forms become

$$X_{i,p}^{n,m} = F(m-n-1, -m-n-1, 1, \beta^2).$$

When n is assumed to be positive, at least one of the first two arguments of the series is always negative, and therefore the series is a polynomial in β^2. For in the first form with $i-p-m > 0$, the second argument is certainly

negative if m is positive; if m is negative, $n+1-m>0$ and the binomial coefficient shows that $i-p-m<n+1-m$, so that the first argument is negative. Similarly when the second form is valid it also is a terminating series. When n is negative one of the known transformations of the hypergeometric series may be necessary to give a finite form. Hence Hansen's coefficients are reduced to the form

$$X_i^{n,m} = (1+\beta^2)^{-n-1} \sum_p J_p(ie) X_{i,p}^{n,m}$$

where $X_{i,p}^{n,m}$ represents, with a simple factor, a hypergeometric polynomial in β^2. This form was first given by Hill.

46. The periodic series in M found above are evidently legitimate Fourier expansions, satisfying the necessary conditions with $e<1$, and as such are convergent. The Bessel's coefficients are given in explicit form by the series (11) which also is at once seen to be absolutely convergent for all values of e. But in practical applications the expansions are generally ordered not as Fourier series in M but as power series in e. Under these circumstances the question of convergence is altered and needs a special investigation. Now

$$E = M + e \sin E$$

considered as an equation in E has *one* root in the interior of a given contour, and any regular function of this root can be expanded by Lagrange's theorem as a power series in e, provided that

$$|e \sin E| < |E - M|$$

at all points of the given contour*. We have then to find a contour with the required property, and to examine its limits.

We are to regard e and M as given real constants. The equation

$$E = M + \rho \cos\chi + \iota\rho \sin\chi$$

where ρ is constant, defines a circular contour. At any point on it

$$\sin E = \sin(M + \rho\cos\chi)\cosh(\rho\sin\chi) + \iota\cos(M+\rho\cos\chi)\sinh(\rho\sin\chi)$$

so that

$$|\sin E|^2 = \sin^2(M+\rho\cos\chi)\cosh^2(\rho\sin\chi) + \cos^2(M+\rho\cos\chi)\sinh^2(\rho\sin\chi)$$
$$= \cosh^2(\rho\sin\chi) - \cos^2(M+\rho\cos\chi)$$

while

$|E - M| = \rho$.

* Cf. Whittaker's *Modern Analysis*, p. 106; Whittaker and Watson, p. 133.

The most unfavourable point on the contour for the required condition is that at which $|\sin E|$ is greatest. And our series is to be valid for all real values of M. Hence the condition is always fulfilled if it is fulfilled when

$$\sin\chi = \pm 1, \quad \cos(M + \rho\cos\chi) = 0$$

or

$$\chi = \pm\tfrac{1}{2}\pi, \quad M = \pm\tfrac{1}{2}\pi$$

in which case

$$|\sin E| = \cosh\rho.$$

Thus the required condition becomes

$$e < \rho/\cosh\rho.$$

The greatest value of e is therefore limited by the maximum value of $\rho/\cosh\rho$, which is given by

$$\cosh\rho = \rho\sinh\rho.$$

Inspection of a table of hyperbolic cosines shows at once that $\rho/\cosh\rho$ is greatest when ρ is about 1·20 and that its value is then about $\frac{2}{3}$. With ordinary logarithmic tables an accurate value can be obtained without difficulty thus. Let $\tan\alpha$ be the greatest possible value of e, so that

$$\tan\alpha = \rho/\cosh\rho = 1/\sinh\rho.$$

It easily follows that

$$\exp.\ \rho = \cot\tfrac{1}{2}\alpha, \quad \coth\rho = \sec\alpha$$

whence, by the equation giving ρ,

$$\cos\alpha \operatorname{Log}\cot\tfrac{1}{2}\alpha = 1$$

or, using common logarithms and taking logarithms once more,

$$\log\cos\alpha + \log\log\cot\tfrac{1}{2}\alpha + 0{\cdot}362\,215\,69 = 0.$$

In this form it is easily verified that

$$\alpha = 33^\circ\,32'\,3''{\cdot}0, \quad \tan\alpha = 0{\cdot}662\,7434\ldots.$$

This last number is then the limiting value of e, within which the expansion of any regular function of E in powers of e is valid for all values of M. The orbits of the members of the solar system have eccentricities which are much below this limit, with the exception of some, but not all, of the periodic comets.

47. In the form in which Bessel's coefficients occur most frequently in astronomical expansions,

$$\frac{2}{e}J_j(je) = \left(\frac{je}{2}\right)^{j-1}\frac{1}{(j-1)!}\left\{1 - \frac{j^2e^2}{2.(2j+2)} + \frac{j^4e^4}{2.4.(2j+2)(2j+4)} - \ldots\right\}$$

$$2J_j'(je) = \left(\frac{je}{2}\right)^{j-1}\frac{1}{(j-1)!}\left\{1 - \frac{j+2}{j}\cdot\frac{j^2e^2}{2.(2j+2)} + \frac{j+4}{j}\cdot\frac{j^4e^4}{2.4.(2j+2)(2j+4)} - \ldots\right\}.$$

It may be convenient for reference to give the following table:

$$\frac{2}{e}J_1(e) = 1 - \frac{e^2}{8} + \frac{e^4}{192} - \frac{e^6}{9216} + \dots$$

$$\frac{2}{e}J_2(2e) = e\left(1 - \frac{e^2}{3} + \frac{e^4}{24} - \frac{e^6}{360} + \dots\right)$$

$$\frac{2}{e}J_3(3e) = \frac{9e^2}{8}\left(1 - \frac{9e^2}{16} + \frac{81e^4}{640} - \dots\right)$$

$$\frac{2}{e}J_4(4e) = \frac{4e^3}{3}\left(1 - \frac{4e^2}{5} + \frac{4e^4}{15} - \dots\right)$$

$$\frac{2}{e}J_5(5e) = \frac{625e^4}{384}\left(1 - \frac{25e^2}{24} + \frac{625e^4}{1344} - \dots\right)$$

$$\frac{2}{e}J_6(6e) = \frac{81e^5}{40}\left(1 - \frac{9e^2}{7} + \frac{81e^4}{112} - \dots\right)$$

....

$$2J_1'(e) = 1 - \frac{3e^2}{8} + \frac{5e^4}{192} - \frac{7e^6}{9216} + \dots$$

$$2J_2'(2e) = e\left(1 - \frac{2e^2}{3} + \frac{e^4}{8} - \frac{e^6}{90} + \dots\right)$$

$$2J_3'(3e) = \frac{9e^2}{8}\left(1 - \frac{15e^2}{16} + \frac{189e^4}{640} - \dots\right)$$

$$2J_4'(4e) = \frac{4e^3}{3}\left(1 - \frac{6e^2}{5} + \frac{8e^4}{15} - \dots\right)$$

$$2J_5'(5e) = \frac{625e^4}{384}\left(1 - \frac{35e^2}{24} + \frac{375e^4}{448} - \dots\right)$$

$$2J_6'(6e) = \frac{81e^5}{40}\left(1 - \frac{12e^2}{7} + \frac{135e^4}{112} - \dots\right)$$

....

These can easily be carried further if necessary, but they are often enough for practical purposes.

Bessel's coefficients occur naturally in several physical problems discussed by Euler and D. Bernoulli from 1732 onwards. In 1771 Lagrange* gave the expression of the eccentric anomaly in terms of the mean anomaly, the result (19) above, and found the expansions of the coefficients as power series, thus anticipating Bessel's work (1824) of more than half a century later.

* *Oeuvres*, III, p. 130. This reference, which seems to have been overlooked, is due to Prof. Whittaker.

CHAPTER V

RELATIONS BETWEEN TWO OR MORE POSITIONS IN AN ORBIT AND THE TIME

48. Since a conic section can be chosen to satisfy any five conditions it is evident that when the focus is given, and two points on the curve, an infinite number of orbits will pass through them. The orbit becomes determinate when the length of the transverse axis is given, though in general the solution is not unique. For let the points be P_1, P_2 and the focal distances r_1, r_2. In the first place we take an elliptic orbit with major axis $2a$. The second focus lies on the circle with centre P_1 and radius $2a - r_1$; it also lies on the circle with radius P_2 and radius $2a - r_2$. These two circles intersect in two points provided (c being the length of the chord P_1P_2)

$$2a - r_1 + 2a - r_2 > c$$

or

$$4a > r_1 + r_2 + c \quad\text{.................................}(1)$$

If this inequality be satisfied two orbits fulfil the given conditions; if not, no such orbit exists. We notice that the two intersections lie on opposite sides of the chord P_1P_2, so that in the one case the two foci lie on the same side of the chord, in the other on opposite sides. In other words, in one orbit the chord intersects the axis at some point between the foci, while in the other orbit it does not. Only when $4a = r_1 + r_2 + c$ the two circles mentioned touch one another in a single point on P_1P_2 and the two orbits coincide. In this case the chord passes through the second focus.

When the orbit is the concave branch of an hyperbola the second focus lies on the circle with centre P_1 and radius $r_1 + 2a$ and also on the circle with centre P_2 and radius $r_2 + 2a$. These circles always intersect in two distinct real points since

$$r_1 + 2a + r_2 + 2a > c$$

always. There are therefore always two hyperbolas which satisfy the conditions. The second foci lie on opposite sides of the chord and hence in the one case the chord intersects the axis between the two foci and the difference

between the true anomalies at the points P_1, P_2 is less than 180°, while in the other case the chord intersects the axis beyond the attracting focus and the difference between the anomalies is greater than 180°.

Under a repulsive force varying inversely as the square of the distance the convex branch of an hyperbola can be described. The position of the second focus is again given by the intersection of two circles, the one with centre P_1 and radius $r_1 - 2a$ and the other with centre P_2 and radius $r_2 - 2a$. These circles intersect in two points provided

$$r_1 - 2a + r_2 - 2a > c$$

or

$$4a < r_1 + r_2 - c \qquad \text{.................................}(2)$$

There are then two hyperbolas and in the one case the chord intersects the axis at a point between the two foci while in the other it cuts the axis at a point beyond the second focus.

It is easy to see similarly that it is always possible to draw four hyperbolas such that one branch passes through P_1 while the other branch passes through P_2. These have no interest from the kinematical point of view since it is impossible for a particle to pass from one branch to the other.

The case of parabolic solutions, two of which always exist, can be inferred from the foregoing by the principle of continuity. But it is otherwise clear that the directrix touches the circles with centres P_1, P_2 and radii r_1, r_2. These circles, which intersect in the focus, have two real common tangents either of which may be the directrix. The corresponding axes are the perpendiculars from the focus to these tangents. In the case of the nearer tangent it is evident that the part of the axis beyond the focus intersects the chord P_1P_2 and the difference of the anomalies is greater than 180°. In the case of the opposite tangent, on the other hand, it is the part of the axis towards the directrix which cuts the chord and the difference of the anomalies is less than 180°.

These simple geometrical considerations show that, when the transverse axis is given, two points on an orbit may be joined in general by four elliptic arcs (of two ellipses), by two concave hyperbolic arcs, by two convex hyperbolic arcs; and in particular by two parabolic arcs. This conclusion is qualified by the conditions (1) and (2) which of course cannot be satisfied simultaneously. All these different cases must present themselves when we seek the time occupied in passing from one given point to another, as we shall at once see.

49. Let E_1, E_2 be the eccentric anomalies at two points P_1, P_2 on an ellipse, and let

$$2G = E_2 + E_1, \quad 2g = E_2 - E_1.$$

Then

$$r_1 = a\,(1 - e\cos E_1), \quad r_2 = a\,(1 - e\cos E_2)$$

and

$$r_1 + r_2 = 2a\{1 - e\cos\tfrac{1}{2}(E_2 + E_1)\cos\tfrac{1}{2}(E_2 - E_1)\}$$
$$= 2a(1 - e\cos G\cos g).$$

Again, c being the chord P_1P_2,

$$c^2 = a^2(\cos E_2 - \cos E_1)^2 + a^2(1 - e^2)(\sin E_2 - \sin E_1)^2$$
$$= 4a^2\sin^2 G\sin^2 g + 4a^2(1 - e^2)\cos^2 G\sin^2 g.$$

Hence if we put

$$\cos h = e\cos G$$

then

$$c^2 = 4a^2\sin^2 g(1 - \cos^2 h)$$

or

$$c = 2a\sin g\sin h$$

and

$$r_1 + r_2 = 2a(1 - \cos g\cos h).$$

If further we now put

$$\epsilon = h + g, \quad \delta = h - g$$

or

$$\epsilon - \delta = E_2 - E_1, \quad \cos\tfrac{1}{2}(\epsilon + \delta) = e\cos\tfrac{1}{2}(E_2 + E_1) \qquad (3)$$

we have

$$r_1 + r_2 + c = 2a\{1 - \cos(h + g)\} = 4a\sin^2\tfrac{1}{2}\epsilon \qquad (4)$$
$$r_1 + r_2 - c = 2a\{1 - \cos(h - g)\} = 4a\sin^2\tfrac{1}{2}\delta \qquad (5)$$

But on the other hand, if $E_2 > E_1$ and

$$\mu = k^2(1 + m) = n^2a^3$$

the time t of describing the arc P_1P_2 is given by

$$nt = E_2 - E_1 - e(\sin E_2 - \sin E_1)$$
$$= \epsilon - \delta - 2\sin\tfrac{1}{2}(\epsilon - \delta)\cos\tfrac{1}{2}(\epsilon + \delta)$$
$$= (\epsilon - \delta) - (\sin\epsilon - \sin\delta) \qquad (6)$$

where ϵ and δ are given by (4) and (5) in terms of $r_1 + r_2$, c and a; and this is Lambert's theorem for elliptic motion.

50. It is evident that (4) and (5) do not give ϵ and δ without ambiguity, and this point must be examined. We suppose always that $E_2 - E_1 < 360°$, i.e. that the arc described is less than a single circuit of the orbit; and we assume that the eccentric anomaly is reckoned from the pericentre in the direction of motion. Now it is consistent with (3) to take $\frac{1}{2}(\epsilon + \delta)$ between 0 and π and we also have $\frac{1}{2}(\epsilon - \delta)$ between the same limits. Hence $\frac{1}{2}\epsilon$ lies between 0 and π and $\frac{1}{2}\delta$ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. But the equation of the chord P_1P_2 referred to the centre of the ellipse shows that it cuts the axis of x in the point

$$x = a\cos\tfrac{1}{2}(E_2 - E_1)/\cos\tfrac{1}{2}(E_2 + E_1), \quad y = 0$$

so that, if Q is this point, A the pericentre and F_1F_2 the foci,

$$\frac{F_1Q}{AQ}=\frac{x-ae}{x-a}=\frac{\cos\frac{1}{2}(\epsilon-\delta)-\cos\frac{1}{2}(\epsilon+\delta)}{\cos\frac{1}{2}(E_2-E_1)-\cos\frac{1}{2}(E_2+E_1)}=\frac{\sin\frac{1}{2}\epsilon\sin\frac{1}{2}\delta}{\sin\frac{1}{2}E_1\sin\frac{1}{2}E_2}$$

$$\frac{F_2Q}{AQ}=\frac{x+ae}{x-a}=\frac{\cos\frac{1}{2}(\epsilon-\delta)+\cos\frac{1}{2}(\epsilon+\delta)}{\cos\frac{1}{2}(E_2-E_1)-\cos\frac{1}{2}(E_2+E_1)}=\frac{\cos\frac{1}{2}\epsilon\cos\frac{1}{2}\delta}{\sin\frac{1}{2}E_1\sin\frac{1}{2}E_2}.$$

Now $\sin\frac{1}{2}\epsilon$ and $\cos\frac{1}{2}\delta$ are always positive. We may also take E_1 less than 2π and $\sin\frac{1}{2}E_1$ positive; then $\sin\frac{1}{2}E_2$ is negative or positive according as the arc includes or does not include the pericentre. In the first equation the left-hand side is negative when the chord intersects the axis between the pericentre and the first (attracting) focus; in the second when the intersection falls between the pericentre and the second focus. Otherwise both members are positive. Hence we see that $\sin\frac{1}{2}\delta$ is positive if (1) the arc contains the pericentre and the chord intersects F_1A, or (2) the arc does not contain the pericentre and the chord does not intersect F_1A; and that $\cos\frac{1}{2}\epsilon$ is positive if (3) the arc contains the pericentre and the chord intersects F_2A, or (4) the arc does not contain the pericentre and the chord does not intersect F_2A. In other words, $\sin\frac{1}{2}\delta$ is positive when the segment formed by the arc and the chord does not contain the first focus, and $\cos\frac{1}{2}\epsilon$ is positive when the segment does not contain the second focus.

Let ϵ_1 and δ_1 be the smallest positive angles which satisfy (4) and (5). The other possible values are $2\pi-\epsilon_1$ and $-\delta_1$. If we put

$$nt_2=\epsilon_1-\sin\epsilon_1,\quad nt_1=\delta_1-\sin\delta_1$$

there are four cases to be distinguished, namely:

(*a*) $$t=t_2-t_1$$

when the segment contains neither focus;

(*b*) $$t=t_2+t_1$$

when the segment contains the attracting, but not the other focus;

(*c*) $$t=2\pi/n-t_2-t_1$$

when the segment contains the second, but not the attracting focus;

(*d*) $$t=2\pi/n-t_2+t$$

when the segment contains both foci. It is easy to see from § 48 that when the extreme points of the arc alone are given these four cases are always presented by the geometrical conditions and can only be distinguished by further knowledge of the circumstances. Usually it is known that the arc is comparatively short and hence that the solution (*a*) is the right one.

51. The corresponding theorem for parabolic motion is easily deduced as a limiting case. For when a is very large ϵ and δ are very small. Hence (4) and (5) become

$$a\epsilon^2 = r_1 + r_2 + c, \quad a\delta^2 = r_1 + r_2 - c.$$

At the same time, if we replace n by $\mu^{\frac{1}{2}}/a^{\frac{3}{2}}$, (6) becomes

$$\mu^{\frac{1}{2}} t = \tfrac{1}{6} a^{\frac{3}{2}} (\epsilon^3 - \delta^3)$$
$$= \tfrac{1}{6}(r_1 + r_2 + c)^{\frac{3}{2}} \mp \tfrac{1}{6}(r_1 + r_2 - c)^{\frac{3}{2}}.$$

As this applies to the motion of a comet, and the mass of a comet may be considered negligible, we may therefore write

$$6kt = (r_1 + r_2 + c)^{\frac{3}{2}} \mp (r_1 + r_2 - c)^{\frac{3}{2}} \quad \text{......(7)}$$

which is the required equation. It was first found by Euler. As regards the ambiguous sign, the second focus is at an infinite distance and does not come into consideration. But δ is negative or positive according as the segment formed by the arc described and the chord contains or does not contain the focus of the parabola. Hence the lower (+) sign is to be used when the angle described by the radius vector exceeds 180°, and the upper (−) sign is to be used when this angle is less than 180°, as it almost always is in actual problems.

52. The solution of (7) as an equation in c is facilitated by a transformation due to Encke. We put

$$c = (r_1 + r_2)\sin\gamma, \quad 0 < \gamma < 90°$$

and

$$\eta = 2kt/(r_1 + r_2)^{\frac{3}{2}}.$$

Then (7) becomes

$$3\eta = (1 + \sin\gamma)^{\frac{3}{2}} \mp (1 - \sin\gamma)^{\frac{3}{2}}$$
$$= (\cos\tfrac{1}{2}\gamma + \sin\tfrac{1}{2}\gamma)^3 \mp (\cos\tfrac{1}{2}\gamma - \sin\tfrac{1}{2}\gamma)^3 \quad \text{......(8)}$$

First we take the upper sign, in which case

$$3\eta = 6\sin\tfrac{1}{2}\gamma\cos^2\tfrac{1}{2}\gamma + 2\sin^3\tfrac{1}{2}\gamma$$
$$= 6\sin\tfrac{1}{2}\gamma - 4\sin^3\tfrac{1}{2}\gamma.$$

If we put

$$\sin\tfrac{1}{2}\gamma = \sqrt{2}\sin\tfrac{1}{3}\Theta, \quad 0 < \tfrac{1}{3}\Theta < 30°$$

then

$$3\eta = 2\sqrt{2}\sin\Theta, \quad 0 < \Theta < 90° \quad \text{......(9)}$$

and

$$\sin\gamma = 2\sqrt{2}\sin\tfrac{1}{3}\Theta\sqrt{(\cos\tfrac{2}{3}\Theta)}.$$

Hence

$$c = (r_1 + r_2)\eta\mu \quad \text{......(10)}$$

where

$$\mu = \sin\gamma/\eta = 3\sin\tfrac{1}{3}\Theta\sqrt{(\cos\tfrac{2}{3}\Theta)}/\sin\Theta \quad \text{......(11)}$$

Since μ and η are both functions of Θ, μ can be tabulated with the argument η. When such a table is available (cf. Bauschinger's *Tafeln*, No. XXII) and η is known, c is immediately given by (10).

In the second place we take the lower sign in (8), so that

$$3\eta = 2\cos^3 \tfrac{1}{2}\gamma + 6 \sin^2 \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma$$

$$= 6 \cos \tfrac{1}{2}\gamma - 4 \cos^3 \tfrac{1}{2}\gamma.$$

If now we put

$$\cos \tfrac{1}{2}\gamma = \sqrt{2} \sin \tfrac{1}{3}\Theta, \quad 30^\circ < \tfrac{1}{3}\Theta < 45^\circ$$

then

$$3\eta = 2\sqrt{2} \sin \Theta, \quad 90^\circ < \Theta < 135^\circ \quad \text{......(12)}$$

and

$$\sin \gamma = 2\sqrt{2} \sin \tfrac{1}{3}\Theta \sqrt{(\cos \tfrac{2}{3}\Theta)}$$

as before. Hence (10) and (11) apply equally to this case, with the difference that Θ as given by (12) is an angle in the second quadrant instead of the first. Except for this the solution is formally the same in both cases, but different tables would be necessary. The case of angular motion exceeding 180°, however, seldom demands consideration in practice.

53. For motion along the concave branch of an hyperbola under attraction to the focus we have (§ 30)

$$r_1 = a(e \cosh E_1 - 1), \quad r_2 = a(e \cosh E_2 - 1)$$

and we may suppose $E_2 > E_1$. Hence

$$r_1 + r_2 = 2a \{e \cosh \tfrac{1}{2}(E_2 - E_1) \cosh \tfrac{1}{2}(E_2 + E_1) - 1\}$$

$$= 2a \{\cosh \tfrac{1}{2}(\epsilon - \delta) \cosh \tfrac{1}{2}(\epsilon + \delta) - 1\}$$

where

$$\epsilon - \delta = E_2 - E_1, \quad \cosh \tfrac{1}{2}(\epsilon + \delta) = e \cosh \tfrac{1}{2}(E_2 + E_1) \quad \text{......(13)}$$

Again, the chord c is given by

$$c^2 = a^2 (\cosh E_2 - \cosh E_1)^2 + a^2 (e^2 - 1)(\sinh E_2 - \sinh E_1)^2$$

$$= 4a^2 \sinh^2 \tfrac{1}{2}(E_2 - E_1) \sinh^2 \tfrac{1}{2}(E_2 + E_1)$$
$$+ 4a^2 (e^2 - 1) \sinh^2 \tfrac{1}{2}(E_2 - E_1) \cosh^2 \tfrac{1}{2}(E_2 + E_1)$$

$$= 4a^2 \sinh^2 \tfrac{1}{2}(\epsilon - \delta) \{-1 + \cosh^2 \tfrac{1}{2}(\epsilon + \delta)\}$$

or

$$c = 2a \sinh \tfrac{1}{2}(\epsilon - \delta) \sinh \tfrac{1}{2}(\epsilon + \delta).$$

Hence

$$r_1 + r_2 + c = 2a(\cosh \epsilon - 1) = 4a \sinh^2 \tfrac{1}{2}\epsilon \quad \text{...............(14)}$$

$$r_1 + r_2 - c = 2a(\cosh \delta - 1) = 4a \sinh^2 \tfrac{1}{2}\delta \quad \text{...............(15)}$$

But on the other hand if

$$\begin{aligned}
\mu &= k^2(1+m) = n^2a^3 \\
nt &= e\sinh E_2 - E_2 - (e\sinh E_1 - E_1) \\
&= 2e\sinh\tfrac{1}{2}(E_2 - E_1)\cosh\tfrac{1}{2}(E_2+E_1) - (E_2 - E_1) \\
&= 2\sinh\tfrac{1}{2}(\epsilon-\delta)\cosh\tfrac{1}{2}(\epsilon+\delta) - (\epsilon-\delta) \\
&= \sinh\epsilon - \sinh\delta - (\epsilon-\delta) \qquad \text{.................(16)}
\end{aligned}$$

where ϵ and δ are given by (14) and (15). This is the form which Lambert's theorem takes in this case.

We may take $\frac{1}{2}(\epsilon+\delta)$ as defined by (13) positive; and $\frac{1}{2}(\epsilon-\delta)$ is positive since $E_2 > E_1$. Hence ϵ is positive. Now the equation of the chord referred to the centre of the hyperbola gives for the intercept on the axis

$$x = -a\cosh\tfrac{1}{2}(E_2-E_1)/\cosh\tfrac{1}{2}(E_2+E_1), \quad y=0$$

or, $(-ae, 0)$ being the attracting focus within this branch,

$$\begin{aligned}
x + ae &= -a\{\cosh\tfrac{1}{2}(\epsilon-\delta) - \cosh\tfrac{1}{2}(\epsilon+\delta)\}/\cosh\tfrac{1}{2}(E_2+E_1) \\
&= +2a\sinh\tfrac{1}{2}\epsilon\sinh\tfrac{1}{2}\delta/\cosh\tfrac{1}{2}(E_2+E_1) \qquad \text{.................(17)}
\end{aligned}$$

The left-hand side is negative or positive according as the intersection falls beyond the focus or on the side of the focus towards the centre. Hence $\sinh\frac{1}{2}\delta$ is positive when the angular motion about the focus is less than 180°, and negative when it exceeds 180°. Thus the sign of δ is determined. If we put

$$m_1^2 = (r_1+r_2+c)/4a, \quad m_2^2 = (r_1+r_2-c)/4a$$

then

$$\sinh\tfrac{1}{2}\epsilon = +m_1, \quad \sinh\tfrac{1}{2}\delta = \pm m_2$$

or

$$\begin{aligned}
\exp.\tfrac{1}{2}\epsilon &= +m_1 + \sqrt{m_1^2+1}, \quad & \exp.\tfrac{1}{2}\delta &= \pm m_2 + \sqrt{m_2^2+1} \\
\sinh\epsilon &= 2m_1\sqrt{m_1^2+1}, \quad & \sinh\delta &= \pm 2m_2\sqrt{m_2^2+1}.
\end{aligned}$$

Hence (16) can be written (Log denoting natural logarithm)

$$\begin{aligned}
nt = {} & 2m_1\sqrt{m_1^2+1} \mp 2m_2\sqrt{m_2^2+1} \\
& - 2\,\mathrm{Log}\,(m_1+\sqrt{m_1^2+1}) \pm 2\,\mathrm{Log}\,(m_2+\sqrt{m_2^2+1})
\end{aligned}$$

where the upper or the lower sign is to be taken according as the angular motion about the attracting focus is less or greater than 180°.

54. The corresponding theorem for motion along the convex branch of an hyperbola under a repulsive force from the focus can be proved similarly. In this case (§ 32)

$$r_1 = a(e\cosh E_1 + 1), \quad r_2 = a(e\cosh E_2 + 1).$$

Hence

$$r_1 + r_2 = 2a\{\cosh\tfrac{1}{2}(\epsilon+\delta)\cosh\tfrac{1}{2}(\epsilon-\delta) + 1\}$$

where

$$\epsilon - \delta = E_2 - E_1, \quad \cosh \tfrac{1}{2}(\epsilon + \delta) = e \cosh \tfrac{1}{2}(E_2 + E_1) \quad \text{.........(18)}$$

and as in § 53

$$c = 2a \sinh \tfrac{1}{2}(\epsilon - \delta) \sinh \tfrac{1}{2}(\epsilon + \delta).$$

We have therefore

$$r_1 + r_2 + c = 2a(\cosh \epsilon + 1) = 4a \cosh^2 \tfrac{1}{2}\epsilon \quad \text{..............(19)}$$

$$r_1 + r_2 - c = 2a(\cosh \delta + 1) = 4a \cosh^2 \tfrac{1}{2}\delta \quad \text{...........(20)}$$

Then by § 32 (22), if $\mu' = n^2a^3$,

$$\begin{aligned} nt &= e \sinh E_2 + E_2 - (e \sinh E_1 + E_1) \\ &= 2e \sinh \tfrac{1}{2}(E_2 - E_1) \cosh \tfrac{1}{2}(E_2 + E_1) + E_2 - E_1 \\ &= 2 \sinh \tfrac{1}{2}(\epsilon - \delta) \cosh \tfrac{1}{2}(\epsilon + \delta) + \epsilon - \delta \\ &= \sinh \epsilon - \sinh \delta + \epsilon - \delta \quad \text{.................................(21)} \end{aligned}$$

where ϵ and δ are given by (19) and (20). This is analogous to the other forms of Lambert's equation.

Putting as before

$$m_1^2 = (r_1 + r_2 + c)/4a, \quad m_2^2 = (r_1 + r_2 - c)/4a$$

we have of necessity

$$\cosh \tfrac{1}{2}\epsilon = + m_1, \quad \cosh \tfrac{1}{2}\delta = + m_2$$

but there is again an ambiguity in the values of ϵ and δ. Now we may take $E_2 > E_1$ and $\frac{1}{2}(\epsilon - \delta)$ positive; and we may define $\frac{1}{2}(\epsilon + \delta)$ as the positive value which satisfies (18). Hence ϵ is positive and exp. $(\frac{1}{2}\epsilon) > 1$. To the equation (17) now corresponds

$$x - ae = -2a \sinh \tfrac{1}{2}\epsilon \sinh \tfrac{1}{2}\delta / \cosh \tfrac{1}{2}(E_2 + E_1)$$

showing that δ is positive if the chord intersects the axis at a point on the side of the focus towards the centre. It must be noticed that this focus is, as before, the focus within the branch and not the centre of force. Hence exp. $\frac{1}{2}\delta >$ or < 1 according as the angular motion about this focus $<$ or $> 180°$. It follows that

$$\text{exp.}\,(\tfrac{1}{2}\epsilon) = + m_1 + \sqrt{m_1^2 - 1}, \quad \text{exp.}\,(\tfrac{1}{2}\delta) = + m_2 \pm \sqrt{m_2^2 - 1}$$

$$\sinh \epsilon = 2m_1\sqrt{m_1^2 - 1}, \quad \sinh \delta = \pm 2m_2\sqrt{m_2^2 - 1}$$

and hence that

$$\begin{aligned} nt = {} & 2m_1\sqrt{m_1^2 - 1} \mp 2m_2\sqrt{m_2^2 - 1} \\ & + 2\,\text{Log}\,(m_1 + \sqrt{m_1^2 - 1}) \mp 2\,\text{Log}\,(m_2 + \sqrt{m_2^2 - 1}) \end{aligned}$$

where Log denotes natural logarithm and the upper or the lower sign is to be taken according as the motion about the internal focus (not the centre of force) is less or greater than 180°.

In all cases, whether the motion is along a parabola or either branch of an hyperbola, when two focal distances are given in position and nothing

more is known about the circumstances, the discussion of § 48 shows that the ambiguities in the expressions for the time of describing the arc correspond to the distinct solutions of the geometrical problem. Hence they cannot be decided without further information. In practice, however, it rarely happens that the angular motion about a focus exceeds 180° and this limitation, by which the upper sign can be taken, will be generally understood.

55. A quantity of great importance in the determination of orbits is the ratio, denoted by y, of the sector to the triangle. The case of elliptic motion is taken first. Since $n = h/ab$, where h is the constant of areas, twice the area of the sector is, by (6),

$$ht = ab\,\{\epsilon - \delta - (\sin\epsilon - \sin\delta)\}.$$

But if (x_1, y_1), (x_2, y_2) are the extremities of the arc, twice the area of the triangle is

$$\begin{aligned}2\Delta &= (x_1 y_2 - x_2 y_1)\\ &= ab\,\{\sin E_2 (\cos E_1 - e) - \sin E_1 (\cos E_2 - e)\}\\ &= ab\,\{\sin (E_2 - E_1) - 2e \cos \tfrac{1}{2}(E_2 + E_1) \sin \tfrac{1}{2}(E_2 - E_1)\}\\ &= ab\,\{\sin(\epsilon - \delta) - (\sin\epsilon - \sin\delta)\}\end{aligned}$$

by (3). Hence

$$y = \frac{\epsilon - \delta - (\sin\epsilon - \sin\delta)}{\sin(\epsilon - \delta) - (\sin\epsilon - \sin\delta)} \quad\text{......................(22)}$$

This expression contains a implicitly and this quantity is to be eliminated. Let $2f$ be the angle between r_1 and r_2 and let g, h have the meaning assigned to them in § 49. Then

$$\begin{aligned}16a^2 \sin^2 \tfrac{1}{2}\epsilon \sin^2 \tfrac{1}{2}\delta &= (r_1 + r_2 + c)(r_1 + r_2 - c)\\ &= (r_1 + r_2)^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos 2f\\ &= 4r_1 r_2 \cos^2 f\end{aligned}$$

whence

$$2a(\cos g - \cos h) = 2\cos f \sqrt{r_1 r_2}.$$

Also by (4) and (5)

$$\begin{aligned}r_1 + r_2 &= 2a(\sin^2 \tfrac{1}{2}\epsilon + \sin^2 \tfrac{1}{2}\delta)\\ &= 2a(1 - \cos g \cos h)\end{aligned}$$

and therefore

$$r_1 + r_2 - 2\cos f \cos g \sqrt{r_1 r_2} = 2a \sin^2 g.$$

Again, by (22),

$$\begin{aligned}y &= \frac{nt}{\sin 2g - 2\sin g \cos h}\\ &= \frac{a\,nt}{\sin g \,.\, 2\cos f \sqrt{r_1 r_2}}.\end{aligned}$$

Hence

$$y^2(r_1+r_2-2\cos f\cos g\sqrt{r_1r_2})=2\mu t^2/(2\cos f\sqrt{r_1r_2})^2 \quad\text{.........(23)}$$

since $n^2a^3=\mu$. On the other hand

$$y-1=\frac{\epsilon-\delta-\sin(\epsilon-\delta)}{\sin(\epsilon-\delta)-(\sin\epsilon-\sin\delta)}$$

$$=\frac{2g-\sin 2g}{2\sin g(\cos g-\cos h)}$$

$$=\frac{a(2g-\sin 2g)}{\sin g\,.\,2\cos f\sqrt{r_1r_2}}$$

and therefore

$$y^2(y-1)=\frac{\mu t^2}{(2\cos f\sqrt{r_1r_2})^3}\cdot\frac{2g-\sin 2g}{\sin^3 g} \quad\text{...............(24)}$$

In the notation of Gauss we write

$$1+2l=\frac{r_1+r_2}{2\cos f\sqrt{r_1r_2}},\quad m^2=\frac{\mu t^2}{(2\cos f\sqrt{r_1r_2})^3}.$$

and then (23) and (24) become

$$y^2=m^2/(l+\sin^2\tfrac{1}{2}g) \quad\text{............................(25)}$$

$$y^3-y^2=m^2(2g-\sin 2g)/\sin^3 g \quad\text{.....................(26)}$$

The value of y is to be found by solving this pair of equations in y and g, the solution being performed by some method of approximation.

56. The corresponding ratio in the case of a parabola can be expressed in several forms. The simplest can be derived as a limiting case from the ellipse when a is large and ϵ and δ are small. For (22) then gives

$$y=\frac{\epsilon^3-\delta^3}{-(\epsilon-\delta)^3+\epsilon^3-\delta^3}=\frac{\epsilon^2+\delta^2+\epsilon\delta}{3\epsilon\delta}.$$

But by §§ 51, 52

$$a^2\epsilon^2\delta^2=(r_1+r_2)^2-c^2=(r_1+r_2)^2\cos^2\gamma.$$

Hence

$$y=\frac{2(r_1+r_2)+(r_1+r_2)\cos\gamma}{3(r_1+r_2)\cos\gamma}$$

$$=\tfrac{1}{3}(1+2\sec\gamma)$$

where

$$c=(r_1+r_2)\sin\gamma.$$

Thus y, like η and μ, is a function of γ (or Θ) and can therefore like μ be tabulated with the argument η, where

$$\eta=2kt/(r_1+r_2)^{\frac{3}{2}}=2\sin\tfrac{1}{2}\gamma(2+\cos\gamma).$$

(Cf. Bauschinger's *Tafeln*, No. XXII *a*.)

57. In the case of the branch of an hyperbola concave to the focus of attraction, twice the area of the sector is by (16)

$$ht = ab\,\{\sinh\epsilon - \sinh\delta - (\epsilon - \delta)\}$$

since $h = \sqrt{(\mu p)} = nab$. And, if (x_1, y_1), (x_2, y_2) are the extremities of the arc, twice the area of the focal triangle is

$$\begin{aligned} 2\Delta &= x_2 y_1 - x_1 y_2 \\ &= ab\,\{\sinh E_1 (\cosh E_2 - e) - \sinh E_2 (\cosh E_1 - e)\} \\ &= ab\,\{\sinh (E_1 - E_2) - e\,(\sinh E_1 - \sinh E_2)\} \\ &= ab\,\{\sinh\epsilon - \sinh\delta - \sinh(\epsilon - \delta)\} \end{aligned}$$

by (13). Hence

$$y = \frac{\sinh\epsilon - \sinh\delta - (\epsilon - \delta)}{\sinh\epsilon - \sinh\delta - \sinh(\epsilon - \delta)} \quad\text{.....................(27)}$$

Now we have by (14) and (15)

$$\begin{aligned} 16a^2 \sinh^2 \tfrac{1}{2}\epsilon \sinh^2 \tfrac{1}{2}\delta &= (r_1 + r_2)^2 - c^2 \\ &= 4 r_1 r_2 \cos^2 f \end{aligned}$$

or

$$2\cos f \sqrt{r_1 r_2} = 2a\,(\cosh h - \cosh g)$$

where $2h = \epsilon + \delta$, $2g = \epsilon - \delta$. Also by addition of the same equations (14) and (15)

$$r_1 + r_2 = 2a\,(\cosh g \cosh h - 1)$$

and therefore

$$r_1 + r_2 - 2\cos f \cosh g \sqrt{r_1 r_2} = 2a \sinh^2 g.$$

But by (27)

$$\begin{aligned} y &= nt/(2\sinh g \cosh h - \sinh 2g) \\ &= a\,nt/\sinh g\,(2\cos f \sqrt{r_1 r_2}) \end{aligned}$$

and therefore

$$y^2 (r_1 + r_2 - 2\cos f \cosh g \sqrt{r_1 r_2}) = 2\mu t^2/(2\cos f \sqrt{r_1 r_2})^2 \quad\text{......(28)}$$

since $n^2 a^3 = \mu$. On the other hand

$$\begin{aligned} y - 1 &= \frac{\sinh(\epsilon - \delta) - (\epsilon - \delta)}{\sinh\epsilon - \sinh\delta - \sinh(\epsilon - \delta)} \\ &= \frac{\sinh 2g - 2g}{2\sinh g\,(\cosh h - \cosh g)} \\ &= \frac{a}{2\cos f \sqrt{r_1 r_2}} \cdot \frac{\sinh 2g - 2g}{\sinh g}. \end{aligned}$$

Hence

$$y^2 (y - 1) = \frac{\mu t^2}{(2\cos f \sqrt{r_1 r_2})^3} \cdot \frac{\sinh 2g - 2g}{(\sinh g)^3} \quad\text{...............(29)}$$

As in the case of the ellipse we write

$$1 + 2l = \frac{r_1 + r_2}{2\cos f \sqrt{r_1 r_2}}, \quad m^2 = \frac{\mu t^2}{(2\cos f \sqrt{r_1 r_2})^3}$$

and thus (28) and (29) become

$$y^2 = m^2/(l - \sinh^2 \tfrac{1}{2}g) \quad\text{.........................(30)}$$

$$y^3 - y^2 = m^2(\sinh 2g - 2g)/\sinh^3 g \quad\text{...............(31)}$$

This pair of equations in y and g must be solved by some process of approximation so that the value of y may be found.

58. The case of the branch which is convex to a centre of repulsive force at the focus $(-ae, 0)$ needs slight modifications. Twice the area of the sector is by (21)

$$ht = ab(\sinh \epsilon - \sinh \delta + \epsilon - \delta)$$

while twice the area of the triangle is

$$\begin{aligned} 2\Delta &= x_1 y_2 - x_2 y_1 \\ &= ab\{\sinh E_2(\cosh E_1 + e) - \sinh E_1(\cosh E_2 + e)\} \\ &= ab\{\sinh(E_2 - E_1) + 2e \sinh \tfrac{1}{2}(E_2 - E_1)\cosh \tfrac{1}{2}(E_2 + E_1)\} \\ &= ab\{\sinh(\epsilon - \delta) + \sinh \epsilon - \sinh \delta\} \end{aligned}$$

by (18). Hence the ratio of sector to triangle is

$$y = \frac{\sinh \epsilon - \sinh \delta + \epsilon - \delta}{\sinh(\epsilon - \delta) + \sinh \epsilon - \sinh \delta} \quad\text{..................(32)}$$

In this case we have by (19) and (20)

$$16a^2 \cosh^2 \tfrac{1}{2}\epsilon \cosh^2 \tfrac{1}{2}\delta = (r_1 + r_2)^2 - c^2 = 4r_1 r_2 \cos^2 f$$

or

$$2\cos f\sqrt{r_1 r_2} = 2a(\cosh h + \cosh g)$$

and

$$r_1 + r_2 = 2a(1 + \cosh h \cosh g)$$

where $2h = \epsilon + \delta$, $2g = \epsilon - \delta$. Hence

$$2\cos f \cosh g\sqrt{r_1 r_2} - (r_1 + r_2) = 2a \sinh^2 g.$$

But (32) may be written

$$\begin{aligned} y &= nt/(\sinh 2g + 2\sinh g \cosh h) \\ &= ant/\sinh g\,(2\cos f\sqrt{r_1 r_2}) \end{aligned}$$

and therefore

$$y^2(2\cos f \cosh g\sqrt{r_1 r_2} - r_1 - r_2) = 2\mu' t^2/(2\cos f\sqrt{r_1 r_2})^2 \quad\text{.........(33)}$$

since $n^2 a^3 = \mu'$. Also by (32)

$$\begin{aligned} 1 - y &= \frac{\sinh(\epsilon - \delta) - (\epsilon - \delta)}{\sinh(\epsilon - \delta) + \sinh \epsilon - \sinh \delta} \\ &= \frac{\sinh 2g - 2g}{2\sinh g(\cosh g + \cosh h)} \\ &= \frac{a}{2\cos f\sqrt{r_1 r_2}} \cdot \frac{\sinh 2g - 2g}{\sinh g}. \end{aligned}$$

Hence

$$y^2(1-y) = \frac{\mu' t^2}{(2\cos f\sqrt{r_1 r_2})^3} \cdot \frac{\sinh 2g - 2g}{\sinh^3 g} \quad \text{..............(34)}$$

If as before we write

$$1 + 2l = \frac{r_1 + r_2}{2\cos f\sqrt{r_1 r_2}}, \quad m^2 = \frac{\mu' t^2}{(2\cos f\sqrt{r_1 r_2})^3}$$

then (33) and (34) become

$$y^2 = m^2/(\cosh^2 \tfrac{1}{2}g - l) \quad \text{..............(35)}$$

$$y^2 - y^3 = m^2(\sinh 2g - 2g)/\sinh^3 g \quad \text{..............(36)}$$

and these again, when solved by a method of approximation, give the value of y in this case when r_1, r_2 and f are known.

59. Some useful approximations can be obtained from a proposition which is easily proved. Let X be any regular function of t. If we neglect powers of t beyond the fourth order we may write

$$X = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$
$$\ddot{X} = \qquad 2a_2 + 6a_3 t + 12 a_4 t^2.$$

Let X_1, X_2, X_3 be the values of X when $t = -\tau_3$, 0 and τ_1. Then we have three pairs of equations, obtained by substituting these values in the above. From these six equations the coefficients $a_0, \ldots, a_4$ can be eliminated and the result expressed in determinant form is clearly

$$\begin{vmatrix} X_1 & 1 & -\tau_3 & \tau_3^2 & -\tau_3^3 & \tau_3^4 \\ X_2 & 1 & 0 & 0 & 0 & 0 \\ X_3 & 1 & \tau_1 & \tau_1^2 & \tau_1^3 & \tau_1^4 \\ \ddot{X}_1 & 0 & 0 & 2 & -6\tau_3 & 12\tau_3^2 \\ \ddot{X}_2 & 0 & 0 & 2 & 0 & 0 \\ \ddot{X}_3 & 0 & 0 & 2 & 6\tau_1 & 12\tau_1^2 \end{vmatrix} = 0.$$

The determinant can be calculated without difficulty, and the result after dividing by $12\tau_1\tau_3(\tau_1+\tau_3)$ is

$$\begin{aligned} 0 = {} & 12X_1\tau_1 + \ddot{X}_1\tau_1(\tau_1^2 - \tau_1\tau_3 - \tau_3^2) \\ & - 12X_2(\tau_1+\tau_3) - \ddot{X}_2(\tau_1+\tau_3)(\tau_1^2 + 3\tau_1\tau_3 + \tau_3^2) \\ & + 12X_3\tau_3 + \ddot{X}_3\tau_3(\tau_3^2 - \tau_1\tau_3 - \tau_1^2). \end{aligned}$$

If we put $\tau_2 = \tau_1 + \tau_3$ and write

$$12A_1 = \tau_2\tau_3 - \tau_1^2, \quad 12A_2 = \tau_1\tau_3 + \tau_2^2, \quad 12A_3 = \tau_1\tau_2 - \tau_3^2 \quad \text{......(37)}$$

this becomes

$$0 = X_1\tau_1\left(1 - \frac{A_1\ddot{X}_1}{X_1}\right) - X_2\tau_2\left(1 + \frac{A_2\ddot{X}_2}{X_2}\right) + X_3\tau_3\left(1 - \frac{A_3\ddot{X}_3}{X_3}\right) \quad \text{...(38)}$$

60. Now in the case of the motion of two bodies in a plane we have

$$\ddot{x} = -\mu x/r^3, \quad \ddot{y} = -\mu y/r^3.$$

Hence substituting x and y successively for X in the formula just obtained we have, to the fourth order in the intervals of time,

$$0 = x_1\tau_1(1 + \mu A_1/r_1^3) - x_2\tau_2(1 - \mu A_2/r_2^3) + x_3\tau_3(1 + \mu A_3/r_3^3)$$
$$0 = y_1\tau_1(1 + \mu A_1/r_1^3) - y_2\tau_2(1 - \mu A_2/r_2^3) + y_3\tau_3(1 + \mu A_3/r_3^3).$$

The solution of these equations in the ordinary form gives

$$\frac{\tau_1(1 + \mu A_1/r_1^3)}{x_2y_3 - x_3y_2} = \frac{\tau_2(1 - \mu A_2/r_2^3)}{-x_3y_1 + x_1y_3} = \frac{\tau_3(1 + \mu A_3/r_3^3)}{x_1y_2 - x_2y_1}.$$

But the denominators are respectively double the areas of the triangles whose sides are pairs of r_1, r_2, r_3. Hence we have the formulae of Gibbs,

$$\frac{[r_2r_3]}{\tau_1(1 + \mu A_1/r_1^3)} = \frac{[r_1r_3]}{\tau_2(1 - \mu A_2/r_2^3)} = \frac{[r_1r_2]}{\tau_3(1 + \mu A_3/r_3^3)} \quad \text{.........(39)}$$

where, according to the customary notation, $[r_2r_3]$ denotes double the area of the triangle whose sides are r_2, r_3, and A_1, A_2, A_3 have the values found above (37). This expresses the ratio of the triangles correctly to the third order of the time intervals.

A second interesting example is provided if we take $X = r^2$. In this case we have (§§ 25 and 26)

$$\frac{d^2}{dt^2}(r^2) = 2\left(\frac{\mu}{r} - \frac{\mu}{a}\right).$$

Hence the formula (38) gives

$$\begin{aligned} r_1^2\tau_1(1 - 2\mu A_1/r_1^3) - r_2^2\tau_2(1 + 2\mu A_2/r_2^3) + r_3^2\tau_3(1 - 2\mu A_3/r_3^3) \\ = -(A_1\tau_1 + A_2\tau_2 + A_3\tau_3)\,2\mu/a \\ = -\{\tau_1(\tau_2\tau_3 - \tau_1^2) + \tau_2(\tau_1\tau_3 + \tau_2^2) + \tau_3(\tau_1\tau_2 - \tau_3^2)\}\,\mu/6a \\ = -(3\tau_1\tau_2\tau_3 - \tau_1^3 + \tau_2^3 - \tau_3^3)\,\mu/6a \\ = -\{3\tau_1\tau_2\tau_3 + 3\tau_1\tau_3(\tau_1 + \tau_3)\}\,\mu/6a \\ = -\mu\tau_1\tau_2\tau_3/a \quad \text{..........(40)} \end{aligned}$$

The form (40) applies to an ellipse and gives the means of calculating an approximate value of a when r_1, r_2, r_3 are known. It must be adapted to the hyperbola by changing the sign of a. For the parabola the right-hand side vanishes and we have the relation between the three radii vectores

$$r_1^2\tau_1 - r_2^2\tau_2 + r_3^2\tau_3 = 2\mu(A_1\tau_1/r_1 + A_2\tau_2/r_2 + A_3\tau_3/r_3)$$

which holds provided we may neglect terms of the fifth order in the time.

61. Returning to the formulae of Gibbs (39), in which the denominators are correct to the fourth order, we have

$$\frac{\tau_1[r_1r_2]}{\tau_3[r_2r_3]} = \frac{1+\mu A_3/r_3^3}{1+\mu A_1/r_1^3} = 1 + \frac{\mu A_3}{r_3^3} - \frac{\mu A_1}{r_1^3}$$

$$\frac{\tau_2[r_1r_2]}{\tau_3[r_1r_3]} = \frac{1+\mu A_3/r_3^3}{1-\mu A_2/r_2^3} = 1 + \frac{\mu A_3}{r_3^3} + \frac{\mu A_2}{r_2^3}$$

$$\frac{\tau_2[r_2r_3]}{\tau_1[r_1r_3]} = \frac{1+\mu A_1/r_1^3}{1-\mu A_2/r_2^3} = 1 + \frac{\mu A_1}{r_1^3} + \frac{\mu A_2}{r_2^3}$$

to the third order. But to the first order

$$\frac{1}{r_3^3} = \frac{1}{r_2^3} - \frac{3\dot{r}_2}{r_2^4}\tau_1$$

$$\frac{1}{r_1^3} = \frac{1}{r_2^3} + \frac{3\dot{r}_2}{r_2^4}\tau_3.$$

Hence

$$\frac{\tau_1[r_1r_2]}{\tau_3[r_2r_3]} = 1 + \frac{\mu(A_3-A_1)}{r_2^3} - \frac{3\mu\dot{r}_2}{r_2^4}(A_3\tau_1 + A_1\tau_3)$$

$$\frac{\tau_2[r_1r_2]}{\tau_3[r_1r_3]} = 1 + \frac{\mu(A_3+A_2)}{r_2^3} - \frac{3\mu\dot{r}_2}{r_2^4}A_3\tau_1$$

$$\frac{\tau_2[r_2r_3]}{\tau_1[r_1r_3]} = 1 + \frac{\mu(A_1+A_2)}{r_2^3} + \frac{3\mu\dot{r}_2}{r_2^4}A_1\tau_3.$$

For the coefficients we easily find from (37)

$$12(A_2+A_3) = \tau_1\tau_3 + \tau_2^2 + \tau_1\tau_2 - \tau_3^2 = 2(\tau_2^2 - \tau_3^2)$$

$$12(A_1+A_2) = \tau_1\tau_3 + \tau_2^2 + \tau_2\tau_3 - \tau_1^2 = 2(\tau_2^2 - \tau_1^2)$$

$$12(A_3\tau_1 + A_1\tau_3) = \tau_1(\tau_1\tau_2 - \tau_3^2) + \tau_3(\tau_2\tau_3 - \tau_1^2) = \tau_1^3 + \tau_3^3$$

and therefore

$$\left.\begin{aligned}
\frac{[r_1r_2]}{[r_2r_3]} &= \frac{\tau_3}{\tau_1}\left\{1 + \frac{\mu}{6r_2^3}(\tau_1^2 - \tau_3^2) - \frac{\mu\dot{r}_2}{4r_2^4}(\tau_1^3 + \tau_3^3)\right\}\\
\frac{[r_1r_2]}{[r_1r_3]} &= \frac{\tau_3}{\tau_2}\left\{1 + \frac{\mu}{6r_2^3}(\tau_2^2 - \tau_3^2) - \frac{\mu\dot{r}_2}{4r_2^4}(\tau_1\tau_2 - \tau_3^2)\,\tau_1\right\}\\
\frac{[r_2r_3]}{[r_1r_3]} &= \frac{\tau_1}{\tau_2}\left\{1 + \frac{\mu}{6r_2^3}(\tau_2^2 - \tau_1^2) + \frac{\mu\dot{r}_2}{4r_2^4}(\tau_2\tau_3 - \tau_1^2)\,\tau_3\right\}
\end{aligned}\right\} \quad\ldots\ldots(41)$$

These formulae are correct to the third order and if the terms involving $\dot{r}_2$ be omitted they express the ratios of the triangles in terms of the single distance r_2 to the second order. Hence their value for the determination of orbits.

62. Without loss of accuracy the ratios can be expressed in terms of the two distances r_1 and r_3 instead of r_2 and $\dot{r}_2$. The forms found by Encke may be derived thus: we have to the first order

$$r_1 = r_2 - \dot{r}_2\tau_3, \quad r_3 = r_2 + \dot{r}_2\tau_1$$

whence

$$r_3 - r_1 = \dot{r}_2\tau_2, \quad r_1 + r_3 = 2r_2 + \dot{r}_2(\tau_1 - \tau_3)$$

and therefore

$$\frac{1}{(r_1 + r_3)^3} = \frac{1}{8r_2^3} - \frac{3\dot{r}_2}{16r_2^4}(\tau_1 - \tau_3)$$

or

$$\frac{1}{r_2^3} = \frac{8}{(r_1 + r_3)^3} + \frac{24(r_3 - r_1)}{(r_1 + r_3)^4} \cdot \frac{\tau_1 - \tau_3}{\tau_2}.$$

In the terms of the third order we have simply

$$\frac{\dot{r}_2}{4r_2^4}\tau_2 = \frac{4(r_3 - r_1)}{(r_1 + r_3)^4}.$$

Hence the ratios of the triangles to the required order become

$$\left.\begin{aligned} \frac{[r_1r_2]}{[r_2r_3]} &= \frac{\tau_3}{\tau_1}\left\{1 + \frac{4\mu}{3(r_1 + r_3)^3}(\tau_1^2 - \tau_3^2) - \frac{4\mu(r_3 - r_1)}{(r_1 + r_3)^4}\tau_1\tau_3\right\} \\ \frac{[r_1r_2]}{[r_1r_3]} &= \frac{\tau_3}{\tau_2}\left\{1 + \frac{4\mu}{3(r_1 + r_3)^3}(\tau_2^2 - \tau_3^2) - \frac{4\mu(r_3 - r_1)}{(r_1 + r_3)^4}\tau_1\tau_3^2/\tau_2\right\} \\ \frac{[r_2r_3]}{[r_1r_3]} &= \frac{\tau_1}{\tau_2}\left\{1 + \frac{4\mu}{3(r_1 + r_3)^3}(\tau_2^2 - \tau_1^2) + \frac{4\mu(r_3 - r_1)}{(r_1 + r_3)^4}\tau_1^2\tau_3/\tau_2\right\} \end{aligned}\right\} \quad \text{...(42)}$$

where, if t_1, t_2, t_3 are the times corresponding to the distances r_1, r_2, r_3,

$$\tau_1 = t_3 - t_2, \quad \tau_2 = t_3 - t_1, \quad \tau_3 = t_2 - t_1.$$

Equivalent but rather simpler expressions in terms of the extreme distances may be obtained by observing that

$$\frac{1}{r_1^3} = \frac{1}{r_2^3} + \frac{3\dot{r}_2}{r_2^4}\tau_3, \quad \frac{1}{r_3^3} = \frac{1}{r_2^3} - \frac{3\dot{r}_2}{r_2^4}\tau_1$$

whence

$$\frac{\tau_2}{r_2^3} = \frac{\tau_1}{r_1^3} + \frac{\tau_3}{r_3^3}, \quad \frac{3\dot{r}_2}{r_2^4}\tau_2 = \frac{1}{r_1^3} - \frac{1}{r_3^3}.$$

By substitution in (41) it is easily found that

$$\left.\begin{aligned} \frac{[r_1r_2]}{[r_2r_3]} &= \frac{\tau_3}{\tau_1}\left\{1 - \frac{\mu}{12r_1^3}(\tau_2\tau_3 - \tau_1^2) + \frac{\mu}{12r_3^3}(\tau_1\tau_2 - \tau_3^2)\right\} \\ \frac{[r_1r_2]}{[r_1r_3]} &= \frac{\tau_3}{\tau_2}\left\{1 + \frac{\mu}{12r_1^3}(\tau_1\tau_3 + \tau_2^2)\frac{\tau_1}{\tau_2} + \frac{\mu}{12r_3^3}\,\frac{\tau_2^3 - \tau_3^3}{\tau_2}\right\} \\ \frac{[r_2r_3]}{[r_1r_3]} &= \frac{\tau_1}{\tau_2}\left\{1 + \frac{\mu}{12r_1^3}\,\frac{\tau_2^3 - \tau_1^3}{\tau_2} + \frac{\mu}{12r_3^3}(\tau_1\tau_3 + \tau_2^2)\frac{\tau_3}{\tau_2}\right\} \end{aligned}\right\} \quad \text{......(43)}$$

From the method by which all the expressions of this kind have been derived it is clear that the results apply equally to all undisturbed orbits, elliptic or hyperbolic.

CHAPTER VI

THE ORBIT IN SPACE

63. Hitherto we have considered the relative motion of two bodies only as referred to axes in the plane in which the motion takes place. It is now necessary to specify the manner in which the motion in space is usually expressed.

We take a sphere of arbitrary unit radius with the Sun at its centre. The ecliptic for a given date is a great circle on this sphere. That hemisphere which contains the North Pole of the Equator may be called the northern hemisphere. On the ecliptic is a fixed point γ which represents the equinoctial point for the given date and from which longitudes are reckoned in a certain direction. The plane of the orbit is also represented by a great circle which intersects the ecliptic in two points. One of these Ω corresponds to the passage of the moving body from the southern to the northern hemisphere and is called the *ascending node*; the other node is called the *descending node.* The longitude of Ω, or $\gamma\Omega$, may be denoted also by Ω: it is an angle which may have any value between 0° and 360°. The angle between the direction of increasing longitudes along the ecliptic and the direction of increasing true anomaly along the orbit is called the *inclination* and may be denoted by i. It is an angle which may lie between 0° and 180°.

Let P be the point on the great circle of the orbit which represents the radius vector through the perihelion and Q any other point on the same great circle representing a radius vector with the true anomaly w, so that $PQ = w$. We may denote the arc ΩP lying between 0° and 360° by ω, so that $\Omega Q = \omega + w$. This angle, reckoned from the ascending node to any point on the plane of the orbit, is called the *argument of the latitude.* It is possible to regard ω as an element of the orbit, but it has been more usual to define the element ϖ, which is called the *longitude of perihelion*, as the sum of the two angles $\Omega + \omega$ although only one of these is measured along the ecliptic. The angle $\varpi + w$ or $\Omega + \omega + w$ is called the *longitude in the orbit.* We have thus defined the three elements, the longitude of the

ascending node, the inclination of the orbit and the longitude of perihelion, required to fix the position of the orbit in space, and with these it is necessary to mention the date of the ecliptic and equinox to which they are referred.

64. The motion must now be definitely related to the time. Let t_0 be an epoch arbitrarily chosen and T the time of perihelion passage. Then, n being the mean motion, the mean anomaly corresponding to the epoch is

$$M_0 = n(t_0 - T).$$

Either M_0 or T might be regarded as an element of the orbit, but in the case of a planetary orbit it is more usual to employ the *mean longitude at the epoch*, ϵ, which is defined as the sum $\varpi + M_0$. Thus at any time t, if $u = \varpi + w$ is the longitude in the orbit and E the eccentric anomaly, the position of the planet is given by

$$\tan \tfrac{1}{2}(u - \varpi) = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \tfrac{1}{2}E$$

where

$$\begin{aligned} E - e \sin E = M &= n(t - T) \\ &= n(t - t_0) + \epsilon - \varpi. \end{aligned}$$

The mean motion and the mean distance are connected by the relation (§ 24)

$$na^{\frac{3}{2}} = \mu^{\frac{1}{2}} = k''(1+m)^{\frac{1}{2}}$$

where m is the mass of the planet (negligible in the case of minor planets). The complete elements can now be enumerated and illustrated by the case of the planet Mars:

		Mars ($m = 1/3\,093\,500$)	
Epoch	t_0	1900 Jan. 0, 0^h G.M.T.	
Mean longitude ...	ϵ	293° 44′ 51″·36	Equinox 1900.0
Longitude of perihelion	ϖ	334 13 6 ·88	
Longitude of node ...	Ω	48 47 9 ·36	
Inclination	i	1 51 1 ·32	
Eccentricity	e	0·093 308 95	
Mean motion	n	1886″·51862	
Log of mean distance	$\log a$	0 ·182 897 033	

The number of independent elements is six, corresponding to the six constants of integration which enter into the solution of the equations of motion, these being in their general form three in number and of the second order.

When the orbit is parabolic the eccentricity is 1 and the mean distance is infinite. The scale of the orbit is indicated by the perihelion distance q and the time of perihelion passage T is given instead of the mean longitude

at a chosen epoch. Thus preliminary parabolic elements of Comet a 1906 (Brooks) are shown as follows:

$$\begin{array}{ll} T & 1905 \text{ Dec. } 22{\cdot}29263 \text{ G.M.T.} \\ \omega & \left.\begin{array}{l} 89^\circ\ 51'\ 53''{\cdot}7 \\ 286\ \ 24\ \ 22\ {\cdot}1 \\ 126\ \ 26\ \ \ 7\ {\cdot}3 \end{array}\right\} 1906.0 \\ \Omega & \\ i & \\ q & 1{\cdot}296318. \end{array}$$

65. If axes $O(x_1, y_1, z_1)$ be taken such that Ox_1 passes through the node, Oy_1 lies in the plane of the orbit, and Oz_1 is in the direction of the N. pole of the orbit, the coordinates of the planet (or comet) are

$$x_1 = r\cos(\omega + w), \quad y_1 = r\sin(\omega + w), \quad z_1 = 0$$

when its true anomaly is w. Let the axes be turned about Ox_1 so that Oy_1 takes the position Oy_2 in the plane of the ecliptic and Oz_2 is directed towards the N. pole of the ecliptic. Then

$$x_2 = x_1, \quad y_2 = y_1 \cos i - z_1 \sin i, \quad z_2 = z_1 \cos i + y_1 \sin i.$$

Next let the axes be turned about Oz_2 so that Ox_3 passes through the equinoctial point and Oy_3 is in longitude 90°. Then

$$x_3 = x_2 \cos\Omega - y_2 \sin\Omega, \quad y_3 = y_2 \cos\Omega + x_2 \sin\Omega, \quad z_3 = z_2.$$

Hence the relations between (x_3, y_3, z_3) and (x_1, y_1, z_1) are given by

	x_1	y_1	z_1
x_3	$\cos\Omega$	$-\cos i \sin\Omega$	$\sin i \sin\Omega$
y_3	$\sin\Omega$	$\cos i \cos\Omega$	$-\sin i \cos\Omega$
z_3	0	$\sin i$	$\cos i.$

This scheme will give the heliocentric ecliptic coordinates of the planet.

It is convenient to write

$$\begin{aligned} \sin a \sin A &= \cos\Omega, & \sin a \cos A &= -\cos i \sin\Omega \\ \sin b' \sin B' &= \sin\Omega, & \sin b' \cos B' &= \cos i \cos\Omega \end{aligned}$$

for then

$$\begin{aligned} x_3 &= r \sin a \sin(A + \omega + w) \\ y_3 &= r \sin b' \sin(B' + \omega + w) \\ z_3 &= r \sin i \sin(\omega + w). \end{aligned}$$

Hence, if R, L_1, B_1 are the geocentric distance, longitude and latitude (the last always a very small angle) of the Sun, which may be taken from the *Nautical Almanac*, and Δ, λ, β are the geocentric distance, longitude and latitude of the planet,

$$\begin{aligned} \Delta \cos\lambda \cos\beta &= R \cos L_1 \cos B_1 + r \sin a \sin(A' + \omega + w) \\ \Delta \sin\lambda \cos\beta &= R \sin L_1 \cos B_1 + r \sin b \sin(B' + \omega + w) \\ \Delta \sin\beta &= R \sin B_1 \qquad\qquad + r \sin i \sin(\omega + w) \end{aligned}$$

whence the geocentric ecliptic coordinates of the planet.

66. Were the elements given with reference to the equator instead of the ecliptic, and this is sometimes done (though not often), the same formulae would give equatorial coordinates with the substitution of R.A. and declination for longitude and latitude. To obtain equatorial coordinates from ecliptic elements another transformation is necessary. Let the last system of axes be turned about Ox_3 so that Oy_3 comes into the plane of the equator and the new axis Oz_4 is directed towards the N. pole of the equator. Then the obliquity of the ecliptic being denoted by ϵ_0,

$$x_4 = x_3, \quad y_4 = y_3 \cos \epsilon_0 - z_3 \sin \epsilon_0, \quad z_4 = z_3 \cos \epsilon_0 + y_3 \sin \epsilon_0.$$

From the above relations between (x_3, y_3, z_3) and (x_1, y_1, z_1) it follows that (x_4, y_4, z_4) and (x_1, y_1, z_1) are related by the scheme:

	x_1	y_1	z_1
x_4	$\sin a \sin A$	$\sin a \cos A$	$\cos a$
y_4	$\sin b \sin B$	$\sin b \cos B$	$\cos b$
z_4	$\sin c \sin C$	$\sin c \cos C$	$\cos c$

where it is easily seen that

$$\begin{aligned}
\sin a \sin A &= \cos \Omega \\
\sin a \cos A &= -\cos i \sin \Omega \\
\cos a &= \sin i \sin \Omega \\
\sin b \sin B &= \cos \epsilon_0 \sin \Omega \\
\sin b \cos B &= \cos \epsilon_0 \cos i \cos \Omega - \sin \epsilon_0 \sin i \\
\cos b &= -\cos \epsilon_0 \sin i \cos \Omega - \sin \epsilon_0 \cos i \\
\sin c \sin C &= \sin \epsilon_0 \sin \Omega \\
\sin c \cos C &= \sin \epsilon_0 \cos i \cos \Omega + \cos \epsilon_0 \sin i \\
\cos c &= -\sin \epsilon_0 \sin i \cos \Omega + \cos \epsilon_0 \cos i.
\end{aligned}$$

The heliocentric equatorial coordinates of the planet now become

$$\begin{aligned}
x_4 &= r \sin a \sin (A + \omega + w) \\
y_4 &= r \sin b \sin (B + \omega + w) \\
z_4 &= r \sin c \sin (C + \omega + w).
\end{aligned}$$

Thus, for example, the above elements for Comet *a* 1906 lead to

$$\begin{aligned}
x_4 &= r\,[9{\cdot}803389] \sin (243^\circ\ 29'\ 42''{\cdot}3 + w) \\
y_4 &= r\,[9{\cdot}999830] \sin (331\ \ 33\ \ 15\ {\cdot}1 + w) \\
z_4 &= r\,[9{\cdot}887772] \sin (\ 60\ \ 14\ \ 19\ {\cdot}5 + w)
\end{aligned}$$

referred to the equator of 1906·0.

Let (x, y, z) be the geocentric equatorial coordinates of the planet and (X, Y, Z) the corresponding geocentric coordinates of the Sun, which may be taken directly from the *Nautical Almanac* or other ephemeris. Thus

$$x = X + x_4, \quad y = Y + y_4, \quad z = Z + z_4.$$

But

$$x = \Delta \cos \alpha \cos \delta, \quad y = \Delta \sin \alpha \cos \delta, \quad z = \Delta \sin \delta$$

where Δ, α, δ are the geocentric distance, right ascension and declination of the planet. These coordinates can therefore be calculated from the equations

$$\begin{aligned}\Delta \cos \alpha \cos \delta &= X + r \sin a \sin (A + \omega + w)\\ \Delta \sin \alpha \cos \delta &= Y + r \sin b \sin (B + \omega + w)\\ \Delta \sin \delta &= Z + r \sin c \sin (C + \omega + w).\end{aligned}$$

This form of equations, introduced by Gauss, is very convenient for the systematic calculation of positions in an orbit.

67. The direct transformation of the elements from one plane of reference to any other may be made as follows. Let γAB represent the first plane of reference, $\gamma_1 AC$ the second plane and BCP the plane of the orbit. The first set of elements are $\gamma B = \Omega$, $BP = \omega$ and $180° - B = i$. The new elements are $\gamma_1 C = \Omega'$, $CP = \omega'$, and $C = i'$. Also the position of the new plane of reference relative to the old may be defined by $\gamma A = \Omega_1$, $A = i_1$ and the arbitrary origin γ_1 by $\gamma_1 A = \Omega_0$. Hence the sides and angles of the triangle ABC are

$$\begin{aligned}a &= \omega - \omega', & b &= \Omega' - \Omega_0, & c &= \Omega - \Omega_1\\ A &= i_1, & B &= 180° - i, & C &= i'.\end{aligned}$$

Now the analogies of Delambre may be written in the single formula, easily remembered,

$$\frac{\sin \{45° \pm (45° - \frac{1}{2} b \mp a)\}}{\sin \{45° \pm (45° - \frac{1}{2} c)\}} = \frac{\sin \{45° \mp (45° - \frac{1}{2} B \pm A)\}}{\cos \{45° \mp (45° - \frac{1}{2} C)\}}$$

where the ambiguities $\pm \mp$ must be read consistently but independently in two sets of three. Hence taking (1) all lower signs, (2) all + signs, (3) all − signs and (4) all upper signs in the above formula, we have

$$\begin{aligned}\sin \tfrac{1}{2} (\Omega' - \Omega_0 + \omega - \omega') \sin \tfrac{1}{2} i' &= \sin \tfrac{1}{2} (\Omega - \Omega_1) \sin \tfrac{1}{2} (i + i_1)\\ \cos \tfrac{1}{2} (\Omega' - \Omega_0 + \omega - \omega') \sin \tfrac{1}{2} i' &= \cos \tfrac{1}{2} (\Omega - \Omega_1) \sin \tfrac{1}{2} (i - i_1)\\ \sin \tfrac{1}{2} (\Omega' - \Omega_0 - \omega + \omega') \cos \tfrac{1}{2} i' &= \sin \tfrac{1}{2} (\Omega - \Omega_1) \cos \tfrac{1}{2} (i + i_1)\\ \cos \tfrac{1}{2} (\Omega' - \Omega_0 - \omega + \omega') \cos \tfrac{1}{2} i' &= \cos \tfrac{1}{2} (\Omega - \Omega_1) \cos \tfrac{1}{2} (i - i_1).\end{aligned}$$

These formulae will serve directly if for example it is required to refer the elements of a minor planet to the plane of Jupiter's orbit instead of to the ecliptic. Or again, if Ω, ω and i are the elements referred to the ecliptic and equinox at the date T and Ω', ω' and i' the elements for the equinox $T + t$, we may put $\Omega_1 = \Pi_1$, $i_1 = \pi_1$ and $\Omega_0 = \Pi_1 + \psi_1$ where ψ_1 is the *general precession*. Hence when these quantities are known the effect of precession is given by

$$\begin{aligned}\tan \tfrac{1}{2} (\Omega' - \Pi_1 - \psi_1 - \Delta\omega) &= \tan \tfrac{1}{2} (\Omega - \Pi_1) \sin \tfrac{1}{2} (i + \pi_1) / \sin \tfrac{1}{2} (i - \pi_1)\\ \tan \tfrac{1}{2} (\Omega' - \Pi_1 - \psi_1 + \Delta\omega) &= \tan \tfrac{1}{2} (\Omega - \Pi_1) \cos \tfrac{1}{2} (i + \pi_1) / \cos \tfrac{1}{2} (i - \pi_1)\end{aligned}$$

where $\Delta\omega = \omega' - \omega$, and (by Napier's analogy involving $B + C$ and A)

$$\tan \tfrac{1}{2}(i - i') = \frac{\cos \frac{1}{2}(\Omega + \Omega' - 2\Pi_1 - \psi_1)}{\cos \frac{1}{2}(\Omega - \Omega' + \psi_1)} \tan \tfrac{1}{2}\pi_1.$$

68. When the interval t is moderately short, however, these rigorous equations for the effect of precession are not required and it is more convenient to use differential formulae. We now consider γAB as the fixed ecliptic of 1850.0 and $\gamma_1 AC$ as a variable ecliptic. Since

$$\cos C = \sin A \sin B \cos c - \cos A \cos B$$

$$-\sin C \,.\, dC = (\cos A \sin B \cos c + \sin A \cos B)\, dA - \sin A \sin B \sin c \,.\, dc$$

$$= \sin C \cos b \,.\, dA - \sin a \sin B \sin C dc$$

or

$$dC = -\cos b \,.\, dA + \sin a \sin B \,.\, dc \quad \text{.......................................(1)}$$

Also, since

$$\sin C \sin b = \sin B \sin c$$

$$\sin C \cos b \,.\, db = \sin B \cos c \,.\, dc - \cos C \sin b \,.\, dC$$

$$= \sin B (\cos c - \cos C \sin a \sin b)\, dc + \cos C \sin b \cos b \,.\, dA$$

or

$$\sin C \,.\, db = \cos C \sin b \,.\, dA + \sin B \cos a \,.\, dc \quad \text{.........................(2)}$$

Similarly, since

$$\sin C \sin a = \sin A \sin c$$

$$\sin C \cos a \,.\, da = \cos A \sin c \,.\, dA + \sin A \cos c \,.\, dc - \cos C \sin a \,.\, dC$$

$$= (\cos A \sin c + \cos C \sin a \cos b)\, dA$$

$$+ (\sin A \cos c - \sin A \cos C \sin a \sin b)\, dc$$

$$= \cos a \sin b \,.\, dA + \sin A \cos a \cos b \,.\, dc$$

or

$$\sin C \,.\, da = \sin b \,.\, dA + \sin A \cos b \,.\, dc \quad \text{...........................(3)}$$

By a slight change of notation we now put Ω_0, ω_0 and i_0 for the elements at $T = 1850.0$, Ω, ω and i for the elements at time $T + t$ (instead of Ω', ω' and i') and define the position of the ecliptic and equinox at $T + t$ relative to those at T by $\Omega_1 = \Pi$, $i_1 = \pi$ and $\Omega_0 = \Pi + \psi$, so that

$$a = \omega_0 - \omega, \quad b = \Omega - \Pi - \psi, \quad c = \Omega_0 - \Pi$$

$$A = \pi, \quad B = 180° - i_0, \quad C = i.$$

Hence by substitution in (1), (2) and (3)

$$di = \quad -\cos(\Omega - \Pi - \psi)\, d\pi - \sin(\omega_0 - \omega) \sin i_0 \,.\, d\Pi$$

$$\sin i \,.\, d(\Omega - \Pi - \psi) = \cos i \sin(\Omega - \Pi - \psi)\, d\pi - \cos(\omega_0 - \omega) \sin i_0 \,.\, d\Pi$$

$$-\sin i \,.\, d\omega = \quad \sin(\Omega - \Pi - \psi)\, d\pi - \cos(\Omega - \Pi - \psi) \sin \pi \,.\, d\Pi.$$

But in the coefficients of $d\Pi$ we may put $i = i_0$, $\omega = \omega_0$ and $\pi = 0$, this being the mutual inclination of the fixed and moving ecliptic. Hence we have simply

$$\begin{aligned} di\,/dt &= -\cos(\Omega - \Pi - \psi)\, d\pi/dt \\ d\Omega/dt &= d\psi/dt + \cot i \sin(\Omega - \Pi - \psi)\, d\pi/dt \\ d\omega\,/dt &= -\operatorname{cosec} i \sin(\Omega - \Pi - \psi)\, d\pi/dt. \end{aligned}$$

These are to be integrated between $t = t_1$ and $t = t_2$, and the coefficients of $d\pi/dt$ are variable with the time. Provided the interval is no more than a few years, it is sufficiently accurate to proceed thus. Writing

$$\begin{aligned} i_2 &= i_1 - (t_2 - t_1)\cos(\Omega - \Pi - \psi)\, d\pi/dt \\ \Omega_2 &= \Omega_1 + (t_2 - t_1)\,\{d\psi/dt + \cot i \sin(\Omega - \Pi - \psi)\, d\pi/dt\} \\ \omega_2 &= \omega_1 - (t_2 - t_1)\operatorname{cosec} i \sin(\Omega - \Pi - \psi)\, d\pi/dt \end{aligned}$$

we take $\Pi + \psi$, $d\pi/dt$ and $d\psi/dt$ from appropriate tables (e.g. Bauschinger's *Tafeln*, No. XXX) with the argument $T + \frac{1}{2}(t_2 + t_1)$. With $\Omega = \Omega_1$ and $i = i_1$ approximate values of Ω_2, i_2 can be obtained and the calculation is then repeated with the corresponding values $\frac{1}{2}(\Omega_1 + \Omega_2)$, $\frac{1}{2}(i_1 + i_2)$ substituted for Ω and i.

69. It is impossible to correct the first observations of a moving body for parallax in the ordinary way because its distance is unknown. But the line of observation intersects the plane of the ecliptic in a certain point, called by Gauss the *locus fictus*, the position of which can be calculated. If the observation is then treated as though made from this point the effect of parallax is allowed for and also the latitude of the Sun.

Let the observation be made at sidereal time T at a place whose geocentric latitude is ϕ. Let α, δ be the observed R.A. and declination, reduced to mean equinox. The geocentric equatorial coordinates of the place of observation are $(\rho\cos\phi\cos T, \rho\cos\phi\sin T, \rho\sin\phi)$, ρ being the Earth's radius at the place, and the corresponding ecliptic coordinates $(\rho h_1, \rho h_2, \rho h_3)$, where

$$\begin{aligned} h_1 &= \cos l \cos b = \cos\phi\cos T \\ h_2 &= \sin l \cos b = \cos\phi\sin T\cos\epsilon_0 + \sin\phi\sin\epsilon_0 \\ h_3 &= \sin b \qquad\;\; = \sin\phi\cos\epsilon_0 - \cos\phi\sin T\sin\epsilon_0 \end{aligned}$$

ϵ_0 being the obliquity of the ecliptic and l, b the longitude and latitude of the Zenith. Similarly

$$\begin{aligned} H_1 &= \cos\lambda\cos\beta = \cos\delta\cos\alpha \\ H_2 &= \sin\lambda\cos\beta = \cos\delta\sin\alpha\cos\epsilon_0 + \sin\delta\sin\epsilon_0 \\ H_3 &= \sin\beta \qquad\;\; = \sin\delta\cos\epsilon_0 - \cos\delta\sin\alpha\sin\epsilon_0 \end{aligned}$$

are the direction cosines of the line of observation, λ, β being the geocentric longitude and latitude of the observed object. The *Nautical Almanac* gives R_1, L_1 and B_1 the geocentric radius vector, longitude and latitude of the Sun.

Hence in heliocentric ecliptic coordinates the equation of the line of observation is

$$\frac{x + R_1 \cos L_1 \cos B_1 - h_1\rho}{H_1} = \frac{y + R_1 \sin L_1 \cos B_1 - h_2\rho}{H_2}$$
$$= \frac{z + R_1 \sin B_1 - h_3\rho}{H_3} = -\Delta$$

where Δ is the distance from the place of observation to the point (x, y, z) positively in the direction away from the object. If then this line intersects the plane of the ecliptic in the point (the locus fictus)

$$x = -R\cos L, \quad y = -R\sin L, \quad z = 0$$
$$\Delta = (h_3\rho - R_1 \sin B_1)/H_3$$
$$-R\cos L = -R_1 \cos L_1 \cos B_1 + \rho h_1 - (h_3\rho - R_1 \sin B_1)\, H_1/H_3$$
$$-R\sin L = -R_1 \sin L_1 \cos B_1 + \rho h_2 - (h_3\rho - R_1 \sin B_1)\, H_2/H_3.$$

But these exact equations can be simplified, regard being had to the small quantities involved. For $B_1 < 1''$ in general, so that $\sin B_1 = B_1$, $\cos B_1 = 1$. Also we may put $\rho = pR_1$ where p is the solar parallax, $8''\cdot80$. Hence writing $R = R_1 + dR_1$, $L = L_1 + dL_1$, we have

$$\Delta = R_1(h_3 p - B_1)/H_3$$
$$-\cos L_1 . dR_1 + R_1 \sin L_1 . dL_1 = pR_1 h_1 - (h_3 p - B_1)\, R_1 H_1/H_3$$
$$-\sin L_1 . dR_1 - R_1 \cos L_1 . dL_1 = pR_1 h_2 - (h_3 p - B_1)\, R_1 H_2/H_3$$

whence

$$-dR_1/R_1 = p(h_1 \cos L_1 + h_2 \sin L_1) - (h_3 p - B_1)(H_1 \cos L_1 + H_2 \sin L_1)/H_3$$
$$dL_1 = p(h_1 \sin L_1 - h_2 \cos L_1) - (h_3 p - B_1)(H_1 \sin L_1 - H_2 \cos L_1)/H_3$$

or again

$$-dR_1/R_1 = p\cos b \cos(L_1 - l) - (p\sin b - B_1)\cos(L_1 - \lambda)\cot\beta$$
$$dL_1 = p\cos b \sin(L_1 - l) - (p\sin b - B_1)\sin(L_1 - \lambda)\cot\beta$$
$$\Delta/R_1 = (p\sin b - B_1)/\sin\beta.$$

Here both p and B_1 are naturally expressed in seconds of arc. Thus dL_1, the additive correction to the Sun's longitude, is appropriately expressed in the same unit. The *Nautical Almanac* gives $\log R_1$, to which the additive correction is

$$d.\log R_1 = \frac{dR_1}{R_1} \cdot \frac{\log_{10}\epsilon}{206265''} = \frac{dR_1}{R_1}[4\cdot3234 - 10].$$

Finally, had the observation actually been made from the locus fictus it would have been made later in time by the interval required for light to travel the distance Δ. But the light equation, or the time over the mean distance from the Sun to the Earth, is $498^s\cdot5$. Hence the additive correction to the time of observation is (in seconds)

$$dt = \frac{\Delta}{R_1} \cdot \frac{498^s\cdot5}{206265''} = \frac{\Delta}{R_1}[7\cdot3832 - 10].$$

The reduction to the locus fictus is a refinement rarely employed in practice.

CHAPTER VII

CONDITIONS FOR THE DETERMINATION OF AN ELLIPTIC ORBIT

70. There are certain properties of the apparent motion of a planet or comet on the celestial sphere which bear on the problem of determining the true orbit and which can be considered with advantage apart from the details of numerical calculation which are necessary for a practical solution. They are closely connected with the direct method of solution devised by Laplace, but they equally contain principles which are fundamental to all methods.

Let (x, y, z) be the heliocentric coordinates of the planet, (X, Y, Z) the heliocentric coordinates of the Earth. Then

$$\ddot{x} = -\mu x/r^3, \ \ldots$$

$$\ddot{X} = -\mu_0 X/R^3, \ \ldots$$

$$\mu = k^2(1+m), \quad \mu_0 = k^2(1+m_0)$$

m and m_0 being the masses of the planet and the Earth. Let (a, b, c) be the corresponding geocentric direction cosines of the planet, so that

$$x = X + a\rho, \quad y = Y + b\rho, \quad z = Z + c\rho \qquad \text{.................(1)}$$

ρ being the geocentric distance of the planet. The observed position of the planet is given in right ascension and declination (α, δ), and if the equatorial system of axes be chosen,

$$a = \cos\alpha\cos\delta, \quad b = \sin\alpha\cos\delta, \quad c = \sin\delta.$$

Since

$$\ddot{x} = \ddot{X} + \ddot{a}\rho + 2\dot{a}\dot{\rho} + a\ddot{\rho}$$

$$\mu x/r^3 - \mu_0 X/R^3 + \ddot{a}\rho + 2\dot{a}\dot{\rho} + a\ddot{\rho} = 0$$

or

$$X(\mu/r^3 - \mu_0/R^3) + \ddot{a}\rho + 2\dot{a}\dot{\rho} + a(\ddot{\rho} + \mu\rho/r^3) = 0$$

and similarly

$$Y(\mu/r^3 - \mu_0/R^3) + \ddot{b}\rho + 2\dot{b}\dot{\rho} + b(\ddot{\rho} + \mu\rho/r^3) = 0$$

$$Z(\mu/r^3 - \mu_0/R^3) + \ddot{c}\rho + 2\dot{c}\dot{\rho} + c(\ddot{\rho} + \mu\rho/r^3) = 0.$$

These are three equations in ρ, $\dot\rho$ and $\ddot\rho+\mu\rho/r^3$, the solution of which can be written down at once in the form

$$\frac{-\rho}{\begin{vmatrix} a & \dot a & X \\ b & \dot b & Y \\ c & \dot c & Z \end{vmatrix}} = \frac{2\dot\rho}{\begin{vmatrix} a & \ddot a & X \\ b & \ddot b & Y \\ c & \ddot c & Z \end{vmatrix}} = \frac{\mu/r^3-\mu_0/R^3}{\begin{vmatrix} a & \dot a & \ddot a \\ b & \dot b & \ddot b \\ c & \dot c & \ddot c \end{vmatrix}} \quad \text{............(2)}$$

the value of $\ddot\rho$ not being required.

71. The determinants in (2) can be calculated when the first and second derivatives of the three direction cosines are known. Now

$$\dot a = -\sin\alpha\cos\delta\,.\,\dot\alpha - \cos\alpha\sin\delta\,.\,\dot\delta$$

$$\ddot a = -\sin\alpha\cos\delta\,.\,\ddot\alpha - \cos\alpha\cos\delta\,.\,\dot\alpha^2 + 2\sin\alpha\sin\delta\,.\,\dot\alpha\dot\delta - \cos\alpha\cos\delta\,.\,\dot\delta^2 - \cos\alpha\sin\delta\,.\,\ddot\delta$$

......

$$\ddot c = \quad \cos\delta\,.\,\ddot\delta - \sin\delta\,.\,\dot\delta^2.$$

The derivatives $\dot\alpha$, $\ddot\alpha$, $\dot\delta$, $\ddot\delta$ are most simply calculated from a series of observed values by Lagrange's interpolation formulae. If the number of observations is three, made at the times t_1, t_2, t_3, we have according to this rule,

$$\alpha = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}\alpha_1 + \frac{(t-t_3)(t-t_1)}{(t_2-t_3)(t_2-t_1)}\alpha_2 + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}\alpha_3$$

whence

$$\dot\alpha = \frac{2t-t_2-t_3}{(t_1-t_2)(t_1-t_3)}\alpha_1 + \frac{2t-t_3-t_1}{(t_2-t_3)(t_2-t_1)}\alpha_2 + \frac{2t-t_1-t_2}{(t_3-t_1)(t_3-t_2)}\alpha_3$$

$$\ddot\alpha = \frac{2\alpha_1}{(t_1-t_2)(t_1-t_3)} + \frac{2\alpha_2}{(t_2-t_3)(t_2-t_1)} + \frac{2\alpha_3}{(t_3-t_1)(t_3-t_2)}$$

or, if we choose $t=t_2$, the time of the middle observation,

$$\alpha = \alpha_2$$

$$\tau_1\tau_2\tau_3\,.\,\dot\alpha = -\tau_1^2\,.\,\alpha_1 + \tau_2(\tau_1-\tau_3)\,.\,\alpha_2 + \tau_3^2\,.\,\alpha_3 = \tau_1^2(\alpha_2-\alpha_1) + \tau_3^2(\alpha_3-\alpha_2)$$

$$\tau_1\tau_2\tau_3\,.\,\ddot\alpha = 2\tau_1\,.\,\alpha_1 - 2\tau_2\,.\,\alpha_2 + 2\tau_3\,.\,\alpha_3 = -2\tau_1(\alpha_2-\alpha_1) + 2\tau_3(\alpha_3-\alpha_2)$$

where

$$\tau_1 = t_3-t_2, \quad \tau_2 = t_3-t_1, \quad \tau_3 = t_2-t_1.$$

These formulae, which apply equally to the declinations, mutatis mutandis, are only correct if the observations are made at very short intervals of time and are ideally accurate. Since the accuracy of observations has practical limitations, moderately long intervals must be used and a greater number of observed places is necessary for satisfactory results. Our immediate concern, however, is rather with general principles than practical methods of calculation.

72. It is now possible to calculate the quantity l given by

$$l = \begin{vmatrix} a & \dot{a} & \ddot{a} \\ b & \dot{b} & \ddot{b} \\ c & \dot{c} & \ddot{c} \end{vmatrix} \div k^2 \begin{vmatrix} a & \dot{a} & X \\ b & \dot{b} & Y \\ c & \dot{c} & Z \end{vmatrix}$$

and we then have by (2)

$$l\rho = (1 + m_0)/R^3 - (1 + m)/r^3 \quad \text{.......................(3)}$$

The mass of the planet, m, must be neglected in a first approximation to the orbit and this is one relation between ρ and r. In essence it is fundamental in all general methods of finding an approximate orbit. A second relation is available because we know the angle ψ between R and ρ, namely

$$r^2 = R^2 + \rho^2 + 2R\rho \cos\psi \quad \text{...........................(4)}$$

while the projection of R as a vector in the direction of ρ gives

$$R \cos\psi = aX + bY + cZ, \quad (0 < \psi < 180°).$$

If r be eliminated between (3) and (4) an equation of the eighth degree in ρ results, and it will be necessary to examine the nature of the possible roots. For the moment we suppose that the appropriate value of ρ has been found. Then the corresponding value of $\dot{\rho}$ is given by (2) and the components of the velocity can be calculated, since by (1)

$$\dot{x} = \dot{X} + \dot{a}\rho + a\dot{\rho}, \quad \dot{y} = \dot{Y} + \dot{b}\rho + b\dot{\rho}, \quad \dot{z} = \dot{Z} + \dot{c}\rho + c\dot{\rho} \quad \text{.........(5)}$$

where $\dot{X}$, $\dot{Y}$, $\dot{Z}$ must be found from the solar ephemeris by mechanical differentiation. Thus when ρ and $\dot{\rho}$ are known, (1) and (5) give the three heliocentric coordinates of the planet and the three corresponding components of velocity at a given time t. From these data the elements of the planet's orbit, assumed for the present purpose to be elliptic, can be calculated without difficulty.

73. Since equatorial coordinates have been used hitherto, the elliptic elements of the orbit will also be referred to the equatorial plane. If new coordinates (ξ, η, ζ) be taken so that the axis of ξ passes through the node and the axis of ζ through the N. pole of the orbit, the transformation scheme is (cf. § 65):

	x	y	z
ξ	$\cos\Omega'$	$\sin\Omega'$	0
η	$-\sin\Omega'\cos i'$	$\cos\Omega'\cos i'$	$\sin i'$
ζ	$\sin\Omega'\sin i'$	$-\cos\Omega'\sin i'$	$\cos i'$

Hence in the plane of the orbit,

$$\zeta = x \sin \Omega' \sin i' - y \cos \Omega' \sin i' + z \cos i' = 0$$

$$\dot{\zeta} = \dot{x} \sin \Omega' \sin i' - \dot{y} \cos \Omega' \sin i' + \dot{z} \cos i' = 0$$

giving for the determination of Ω' and i'

$$\frac{\sin \Omega' \sin i'}{y\dot{z} - \dot{y}z} = \frac{\cos \Omega' \sin i'}{x\dot{z} - \dot{x}z} = \frac{\cos i'}{x\dot{y} - \dot{x}y} \quad \text{.................(6)}$$

Also, if u is the argument of latitude (or rather of declination),

$$\xi = r \cos u = x \cos \Omega' + y \sin \Omega' \quad \text{..............................(7)}$$

and

$$\eta = -x \sin \Omega' \cos i' + y \cos \Omega' \cos i' + z \sin i'$$

or

$$r \sin u = z \operatorname{cosec} i' \quad \text{..(8)}$$

by the above equation for ζ. Similarly, if V is the velocity and χ the angle between V and the radius vector produced,

$$\dot{\xi} = V \cos(u + \chi) = \dot{x} \cos \Omega' + \dot{y} \sin \Omega' \quad \text{..................(9)}$$

$$\dot{\eta} = V \sin(u + \chi) = \dot{z} \operatorname{cosec} i' \quad \text{..........................(10)}$$

Thus V and χ, as well as r and u, are determined. Now if w is the true anomaly at the point, the polar equation of the orbit gives

$$p = r(1 + e \cos w) \quad \text{.........................(11)}$$

$$p \cot \chi = re \sin w \quad \text{...................................(12)}$$

since $\tan \chi = r\,dw/dr$. But the constant of areas is

$$h = Vr \sin \chi = \sqrt{(\mu p)} = k\sqrt{p} \quad \text{.....................(13)}$$

giving p and hence e and w. The mean distance a can be deduced from the known values of p and e, or directly from the relation

$$V^2 = 2\mu/r - \mu/a \quad \text{..............................(14)}$$

and the mean motion n from the equation $\mu = k^2 = n^2 a^3$. Also the element ϖ' is given by $\varpi' = \Omega' + u - w$. Finally the epoch of perihelion passage is determined by the two equations

$$\tan \tfrac{1}{2} E = \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \tfrac{1}{2} w$$

$$n(t - T) = E - e \sin E \quad \text{..........................(15)}$$

E being the eccentric anomaly at the point of the orbit observed.

74. We now return to the consideration of the solution of equations (3) and (4), following the method of Charlier, which gives the clearest view of the geometrical conditions of the problem. The first of these equations is based on the assumption that the point of observation is moving under gravity about the Sun. The point which so moves is in reality the centre

of gravity of the Earth-Moon system and, strictly speaking, the observations should be reduced to this point and not the centre of the Earth. But this is a matter of detail which our immediate purpose does not require us to stop and consider. Similarly we may neglect the mass of the Earth as well as that of the planet and put $R = 1$. Then the equations become simply

$$l\rho = 1 - 1/r^3 \qquad \text{.........(16)}$$

$$r^2 = 1 + 2\rho\cos\psi + \rho^2 \qquad \text{.........(17)}$$

where l and ψ are known. The position of the planet becomes known when either ρ or r has been found, and it is simpler to eliminate ρ. Thus

$$l^2r^8 = l^2r^6 + 2lr^3(r^3 - 1)\cos\psi + (r^3 - 1)^2$$

or

$$l^2r^8 - (l^2 + 2l\cos\psi + 1)\,r^6 + 2(l\cos\psi + 1)\,r^3 - 1 = 0 \quad \text{......(18)}$$

Now the coefficient of r^3 is

$$2(l\cos\psi + 1) = \{(1 - 1/r^3)(r^2 - 1 - \rho^2) + 2\rho^2\}/\rho^2$$
$$= \{(1 - 1/r^3)(r^2 - 1) + \rho^2(1 + 1/r^3)\}/\rho^2$$

which is obviously positive, whether r is greater or less than 1. And the coefficient of r^6 is essentially negative. Hence, by Descartes' rule of signs, there are at most three positive roots and one negative root. The latter certainly exists because the last term is negative (the equation being of even degree), and two positive roots must satisfy the equation, namely $+1$ (corresponding to the Earth's orbit) and the root required. There must be a fourth real root, and therefore in all three real and positive roots, one real and negative root and four imaginary roots. But the third positive root may or may not satisfy the problem.

Now by (16) r is greater or less than 1 according as l is positive or negative. If then the two roots which are in question lie on opposite sides of 1, the spurious root can be detected and a unique solution of the problem can be found. But if they lie on the same side, they cannot be discriminated between in this way, and an ambiguity exists. If we divide (18) by $(r - 1)$, we obtain

$$f(r) = l^2r^6(r + 1) - (2lr^3\cos\psi + r^3 - 1)(r^2 + r + 1) = 0.$$

Thus

$$f(0) = +1, \quad f(+1) = 2l(l - 3\cos\psi)$$

so that the roots are separated by $+1$, and a unique solution exists, if $l(l - 3\cos\psi)$ is negative.

75. The geometrical interpretation is instructive. The equation (16) for different values of the parameter l represents a family of curves in bipolar coordinates, the poles being E (the Earth) for ρ and S (the Sun) for r. The planet lies at the intersection of one of these curves with a straight line

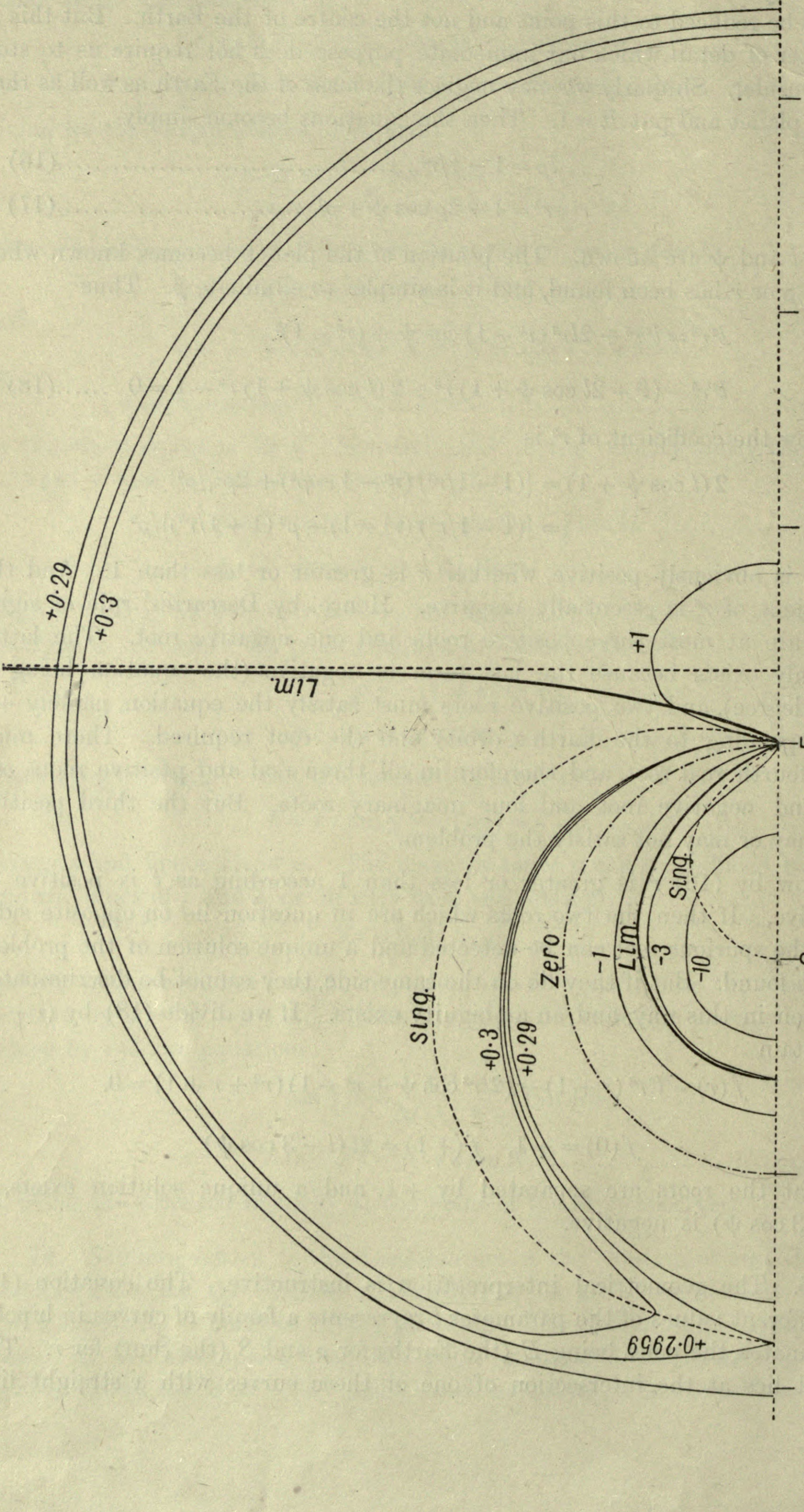


Fig. 3.

drawn through E in a given direction. But there may be two intersections, and this will happen if $f(+1)$ or

$$\rho^2 l(l-3\cos\psi) = (1-1/r^3)\{1-1/r^3+\tfrac{3}{2}(1+\rho^2-r^2)\}$$

is positive. This expression changes sign when we cross the circle $r=1$ and again when we cross the curve

$$1-1/r^3+\tfrac{3}{2}(1+\rho^2-r^2)=0.$$

Putting $\rho^2 = 1+r^2-2r\cos\phi$ we get for the polar equation of this curve with the origin at S

$$4-3r\cos\phi = 1/r^3 \qquad \text{(19)}$$

or in rectangular coordinates,

$$r^3(4-3x)=1$$

showing that the curve has an asymptote $3x=4$. Moving the origin to E we find at once that E is a node, the tangents being $y=\pm 2x$. The whole curve consists of a loop crossing the SE axis at the point $r=\cdot5604$, $\phi=\pi$, and an asymptotic branch, and is shown as the "limiting" curve in the figure. The plane of the figure is that containing S, E and P (the planet); it is only necessary to show the curves on one side of the axis because this is one of symmetry.

A few curves of the family (16) are also shown in the figure, for values of l which indicate sufficiently the different forms. When $l=0$ we have the circle $r=1$, called here the "zero" circle. It is evident that when l is negative $r<1$ and the curve lies entirely within the zero circle, while when l is positive $r>1$ and the curve lies entirely outside this circle. When l has a large negative value, the curve consists of a simple loop surrounding S and an isolated conjugate point at E. As $-l$ decreases from ∞ the loop increases in size until, when $l=-3$, the loop extends to E, where there is a cusp. Afterwards as l approaches 0 the loop, still passing through E, approximates more and more closely to the zero circle.

When l is positive the form of the curves is rather more complicated. It must be remarked that l cannot be greater than $+3$. For

$$l=(r^3-1)/r^3\rho = (r^{-1}+r^{-2}+r^{-3})(r-1)/\rho.$$

But $r>1$ and $r-1<\rho$. Hence the limit is established and we have only to follow the values of l from $+3$ to 0. At first the curve consists of a small loop passing through E. As the value of l falls the loop expands, tending to enfold the zero circle. Finally, when $l=+0\cdot2959$, it reaches the axis again and forms a node on the further side of S. As the value of l falls still further the curve breaks up into two distinct loops. The larger continues to expand outwards at all points and recedes to infinity, while the inner, always passing through E, contracts until finally it becomes the zero circle. These features in the development of the family of curves will be evident in the figure.

It will now be apparent that the limiting curve and the zero circle divide space into certain regions and that the solution of the problem of determining an orbit by the method indicated is unique or not according to the region in which the planet happens to be. Thus we distinguish four cases:

(1) If the planet is within the loop of the limiting curve there are two solutions.

(2) In the space between the loop and the zero circle the solution is unique.

(3) Outside the zero circle and to the left of the asymptotic branch of the limiting curve there are again two solutions.

(4) If the planet lies to the right of the asymptotic branch of the limiting curve only one solution is possible. It happens that newly discovered minor planets are usually observed near opposition and therefore this is the case which most commonly occurs.

76. There is another curve which has considerable importance in the problem of determining an orbit by a method of approximation and to which Charlier has given the name of the "singular" curve. We may find it thus. If we eliminate r between the equations (16) and (17) we have

$$l\rho = 1 - (1 + 2\rho \cos \psi + \rho^2)^{-\frac{3}{2}}$$

which is an equation giving the values of ρ for a line drawn through E in the direction ψ. Two of the values become equal and the line touches the curve (16) if

$$\begin{aligned} l &= 3 (\cos \psi + \rho) (1 + 2\rho \cos \psi + \rho^2)^{-\frac{5}{2}} \\ &= 3 (\cos \psi + \rho)/r^5. \end{aligned}$$

Hence the locus of the points of contact of the tangents from E to the family of curves (16) is

$$(1 - 1/r^3)/\rho = 3 (\cos \psi + \rho)/r^5$$

or

$$2r^2 (r^3 - 1) = 3 (\rho^2 + r^2 - 1)$$

or again

$3(\rho^2-1) = 2r^5 - 5r^2$

$$3\rho^2 = 2r^5 - 5r^2 + 3 \quad\text{.........................(20)}$$

This is the equation of the singular curve. If we change from bipolar coordinates to the polar equation with the origin at S, we obtain

$$3 (1 - 2r \cos \phi + r^2) = 2r^5 - 5r^2 + 3$$

or

$$r^3 = 4 - 3 \cos \phi/r \quad\text{..............................(21)}$$

Comparison of this form with the equation (19) of the limiting curve shows at once that these two curves are the inverse of one another with respect to

the zero circle. From this relation the form of the singular curve, which is shown in figure 3, becomes apparent.

The importance of the singular curve arises thus. In general a line through E meets a curve of the family (16) either in one point (besides E) or in two distinct points. In the latter case the coordinates of the planet are regular functions of the time and can be expanded in powers of the time, but each is expressed by two distinct series between which it is impossible to discriminate. When, however, the planet is situated at a point on the singular curve, the two distinct series coalesce and each point of the singular curve corresponds to a branch point where we may expect the coordinates of the planet to be no longer regular functions of the time. This is in fact the case. Charlier obtained the equation of the singular curve by noticing that along this curve expansion of the coordinates as power series in the time ceases to be possible.

77. If the masses of the Earth and of the planet be neglected, (2) may be written in the form

$$\frac{-\rho}{\Delta_1} = \frac{2\dot{\rho}}{\Delta_2} = \frac{k^2(1/r^3 - 1/R^3)}{\Delta_3} \quad \text{.........................(22)}$$

where Δ_1, Δ_2, Δ_3 represent three determinants and $l = \Delta_3/k^2\Delta_1$. It is clear, as we have already noticed, that $r < R$ if l is negative and $r > R$ if l is positive. Now the equation of the plane of the great circle tangent to the apparent orbit at (a, b, c) is

$$\begin{vmatrix} a & \dot{a} & x \\ b & \dot{b} & y \\ c & \dot{c} & z \end{vmatrix} = 0 \quad \text{.............................(23)}$$

The coordinates of the Sun on the celestial sphere are $(-X/R, -Y/R, -Z/R)$ and of a neighbouring point to (a, b, c) on the apparent orbit $(a + \dot{a}t + \frac{1}{2}\ddot{a}t^2$, $b + \ldots, c + \ldots)$. Hence the ratio of the perpendiculars from these points to the above plane is $-\Delta_1/R \div \frac{1}{2}t^2\Delta_3 = -2/lk^2t^2R$. Thus l is negative if the Sun and the arc of the planet's orbit lie on the same side of the great circle touching the orbit, and positive if the Sun and the arc are on opposite sides. In the first case $r < R$, in the second $r > R$. Hence we have the theorem due to Lambert, which may be expressed by saying that an arc of the orbit of an inferior planet appears concave to the corresponding position of the Sun, but the arc described by a superior planet appears convex. This test makes it immediately apparent whether a planet or the Earth is the nearer to the Sun.

It may happen that Δ_3 vanishes. It is then necessary to express the coordinates of neighbouring points on the orbit to the third order

$(a \pm \dot{a}t + \frac{1}{2}\ddot{a}t^2 \pm \frac{1}{6}\dddot{a}t^3,\ b \pm \ldots,\ c \pm \ldots)$. The result of substituting in the left-hand side of (23) is

$$\pm \tfrac{1}{6}t^3 \begin{vmatrix} a & \dot{a} & \dddot{a} \\ b & \dot{b} & \dddot{b} \\ c & \dot{c} & \dddot{c} \end{vmatrix}$$

and the double sign shows that the curve crosses the tangent great circle. In the language of plane geometry there is a point of inflexion on the apparent orbit. Now if Δ_3 vanishes either $r = R$ or $\Delta_1 = 0$. Thus such a point of inflexion occurs either when a comet reaches the same distance from the Sun as the Earth or when the great circle which touches the orbit of a planet passes through the position of the Sun.

78. When the apparent orbit of a planet reaches a stationary point the curve either crosses itself and forms a loop, or without crossing itself it pursues a twisted path, passing through a point of inflexion. At such a point, as we have just seen, the tangent in general passes through the Sun. There is a related theorem, due to Klinkerfues, which applies to the case of a loop. Let P_1, P_2, P_3 be three positions of the planet in space, E_1, E_2, E_3 the corresponding positions of the Earth and S the position of the Sun. If the first and third positions correspond to the double point on the loop, E_1P_1 and E_3P_3 are parallel and lie in one plane. Let SP_2 meet the chord P_1P_3 in p_2 and SE_2 meet the chord E_1E_3 in e_2. If t_1 is the time taken to describe P_1P_2 or E_1E_2 and t_2 the time along P_2P_3 or E_2E_3, $t_1 : t_2$ is the ratio of the sectors SP_1P_2, SP_2P_3 or very nearly the ratio of the triangles SP_1p_2, Sp_2P_3, that is $P_1p_2 : p_2P_3$. But similarly $t_1 : t_2$ is nearly equal to the ratio $E_1e_2 : e_2E_3$. Hence P_1P_3 and E_1E_3 are divided by p_2 and e_2 in approximately the same ratio and therefore e_2p_2 is parallel to E_1P_1 and E_3P_3. Consequently the three planes E_1SP_1, $E_2e_2Sp_2P_2$, E_3SP_3 have a common line of intersection, namely the line through S parallel to E_1P_1 and E_3P_3. But on the geocentric sphere these three planes correspond to three intersecting great circles. The first and third intersect in P, the double point on the apparent orbit. Hence the great circle joining any intermediate point on the loop to the corresponding position of the Sun also passes through the double point, at least very approximately.

It may be inferred then that if any three points on such a loop be joined to the corresponding positions of the Sun, the three great circles will meet in one point which is also a point on the apparent orbit.

79. There is some interest in finding the geometrical meaning of the three determinants Δ_1, Δ_2, Δ_3 in (2) or (22). Bruns has noticed that $\Delta_3 = V^3k$, where k is the geodetic curvature of the apparent orbit on the sphere and V the velocity in this orbit at the point (a, b, c), so that

$$V^2 = \dot{a}^2 + \dot{b}^2 + \dot{c}^2.$$

But we shall now express these determinants in terms of the small circle of closest contact or circle of curvature. This passes through the points (a, b, c), $(a+\dot{a}t, b+\dot{b}t, c+\dot{c}t)$ and $(a+\dot{a}t'+\frac{1}{2}\ddot{a}t'^2, b+\ldots, c+\ldots)$, and the equation of its plane is

$$\begin{vmatrix} x & y & z & 1 \\ a & b & c & 1 \\ \dot{a} & \dot{b} & \dot{c} & 0 \\ \ddot{a} & \ddot{b} & \ddot{c} & 0 \end{vmatrix} = 0$$

or

$$x(\dot{b}\ddot{c} - \ddot{b}\dot{c}) + y(\dot{c}\ddot{a} - \ddot{c}\dot{a}) + z(\dot{a}\ddot{b} - \ddot{a}\dot{b}) = \Delta_3 \qquad \text{.............. (24)}$$

Now

$$a^2 + b^2 + c^2 = 1$$
$$a\dot{a} + b\dot{b} + c\dot{c} = 0$$
$$a\ddot{a} + b\ddot{b} + c\ddot{c} = -V^2$$

by successive differentiation. Solving these as linear equations in a, b, c, we obtain

$$a\Delta_3 = \dot{b}\ddot{c} - \ddot{b}\dot{c} - V^2(b\dot{c} - \dot{b}c)$$

and two similar equations. But $(\dot{a}/V, \dot{b}/V, \dot{c}/V)$ are the direction cosines of the point P_1 on the tangent 90° from (a, b, c), and the pole of the tangent is (a_0, b_0, c_0) where

$$Va_0 = b\dot{c} - \dot{b}c, \quad Vb_0 = c\dot{a} - \dot{c}a, \quad Vc_0 = a\dot{b} - \dot{a}b$$

so that

$$\dot{b}\ddot{c} - \ddot{b}\dot{c} = a\Delta_3 + V^3 a_0, \ldots$$

and

$$\Sigma(\dot{b}\ddot{c} - \ddot{b}\dot{c})^2 = \Delta_3^2 + V^6.$$

The equation of the circle of curvature (24) becomes then

$$(a\Delta_3 + a_0V^3)x + (b\Delta_3 + b_0V^3)y + (c\Delta_3 + c_0V^3)z = \Delta_3.$$

Hence, if ω is the angular radius of this circle,

$$\cos^2\omega = \Delta_3^2/(\Delta_3^2 + V^6)$$

and therefore

$$\Delta_3 = V^3 \cot\omega.$$

This then is the geometrical meaning of the third determinant.

80. Next we take Δ_2. If (A, B, C) are the geocentric direction cosines of the Sun, $X = -AR$, $Y = -BR$, $Z = -CR$ and

$$\begin{aligned}\Delta_2 &= -R\{A(b\ddot{c} - \ddot{b}c) + B(c\ddot{a} - \ddot{c}a) + C(a\ddot{b} - \ddot{a}b)\} \\ &= -R\frac{d}{dt}\{A(b\dot{c} - \dot{b}c) + B(c\dot{a} - \dot{c}a) + C(a\dot{b} - \dot{a}b)\} \\ &= -R\frac{d}{dt}\{V(Aa_0 + Bb_0 + Cc_0)\} \\ &= -R\dot{V}(Aa_0 + Bb_0 + Cc_0) - RV(A\dot{a}_0 + B\dot{b}_0 + C\dot{c}_0).\end{aligned}$$

Here A, B, C are of course constants. Now (a_0, b_0, c_0) is the pole P_0 of the tangent at P, (a, b, c). The arc PP_0 passes through the centre of the circle of curvature and while P is initially describing a circle of angular radius ω about this centre P_0 is describing a circle of radius $90° - \omega$ about the same centre. If the velocity of P_0, which is in the direction of the pole of PP_0 opposite P_1, is V',

$$V'/\cos\omega = V/\sin\omega, \quad \dot{a}_0/V' = -\dot{a}/V, \quad \dot{b}_0/V' = -\dot{b}/V, \quad \dot{c}_0/V' = -\dot{c}/V.$$

Hence

$$\Delta_2 = \Delta_1 \dot{V}/V + RV\cot\omega\,(A\dot{a} + B\dot{b} + C\dot{c}).$$

Again

$$\begin{aligned}\Delta_1 &= -RV(Aa_0 + Bb_0 + Cc_0)\\ &= -RV\cos SP_0 = -RV\sin\tau\end{aligned}$$

S being the position of the Sun on the sphere, and τ the perpendicular arc from S to the tangent PP_1 at P to the apparent orbit (positive if drawn from the same side of PP_1 as P_0 or the centre of curvature). Also

$$A\dot{a} + B\dot{b} + C\dot{c} = V\cos SP_1 = V\sin\nu$$

where ν is the perpendicular arc from S to the normal PP_0 to the apparent orbit at P (positive if drawn from the same side of PP_0 as P_1). Hence

$$\Delta_2 = -R\dot{V}\sin\tau + RV^2\cot\omega\sin\nu.$$

Thus the geometrical significance of the three determinants has been determined and we may write (2) in the form

$$\frac{\rho}{RV\sin\tau} = \frac{2\dot{\rho}}{R(V^2\cot\omega\sin\nu - \dot{V}\sin\tau)} = \frac{\mu/r^3 - \mu_0/R^3}{V^3\cot\omega}$$

which shows in the clearest way how this method of determining the orbit depends on a knowledge of the simple quantities V, $\dot{V}$, τ, ν and ω, which can be specified without reference to any particular axes. To these must be joined the equation (4), which enjoys the same property.

It has been remarked (§ 75) that l cannot be greater than $+3$. Now

$$l = \Delta_3/k^2\Delta_1 = -V^2\cot\omega/k^2R\sin\tau.$$

Hence for a superior planet,

$$V^2 < 3k^2R\,|\tan\omega\sin\tau|$$

which sets a limit to the apparent velocity when ω and τ are known, or to the curvature of the path when V and τ are known.

CHAPTER VIII

DETERMINATION OF AN ORBIT. METHOD OF GAUSS

81. Since a planetary orbit requires for its complete specification six elements, it is to be expected that three positions of the planet, i.e.. three pairs of coordinates, observed at known times, will suffice to determine its path. And this is in general true, though there are exceptional circumstances in which further observations may be necessary. The formulae are a little simpler when ecliptic coordinates are employed, and though this is not essential we shall take as the data of the problem:

the times of observation	t_1, t_2, t_3
the longitudes of the planet	$\lambda_1, \lambda_2, \lambda_3$
the latitudes of the planet	$\beta_1, \beta_2, \beta_3$
the longitudes of the Earth	L_1, L_2, L_3
the Earth's radii vectores	R_1, R_2, R_3.

The angular coordinates are referred to a fixed equinox which will apply to the resulting elements. The Earth's longitude (which differs by 180° from the Sun's longitude) and radius vector can be derived from the *Nautical Almanac* or other national ephemeris: the Earth's latitude can be neglected, or, if desired, allowed for by using the method of the locus fictus (§ 69).

At the time t_i let r_i be the heliocentric distance of the planet and ρ_i its geocentric distance. Referred to a fixed system of rectangular axes through the Sun let (x_i, y_i, z_i) be the coordinates of the planet, (A_i, B_i, C_i) the direction cosines of R_i and (a_i, b_i, c_i) the direction cosines of ρ_i, so that

$$x_i = a_i\rho_i + A_iR_i, \quad y_i = b_i\rho_i + B_iR_i, \quad z_i = c_i\rho_i + C_iR_i.$$

82. Since the three positions of the planet lie in a plane passing through the Sun

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

or

$$x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1) = 0.$$

But $(y_2z_3 - y_3z_2)$, $(y_1z_3 - y_3z_1)$ and $(y_1z_2 - y_2z_1)$ are the projections on the yz plane of the areas $[r_2r_3]$, $[r_3r_1]$ and $[r_1r_2]$. Hence

$$x_1[r_2r_3] - x_2[r_1r_3] + x_3[r_1r_2] = 0$$

or

$$[r_2r_3](a_1\rho_1 + A_1R_1) - [r_1r_3](a_2\rho_2 + A_2R_2) + [r_1r_2](a_3\rho_3 + A_3R_3) = 0 \ldots (1)$$

And similarly

$$[r_2r_3](b_1\rho_1 + B_1R_1) - [r_1r_3](b_2\rho_2 + B_2R_2) + [r_1r_2](b_3\rho_3 + B_3R_3) = 0 \ldots (2)$$

$$[r_2r_3](c_1\rho_1 + C_1R_1) - [r_1r_3](c_2\rho_2 + C_2R_2) + [r_1r_2](c_3\rho_3 + C_3R_3) = 0 \ldots (3)$$

These are the fundamental equations expressing the condition for a plane orbit. From them one pair of the six quantities ρ_i, R_i can be eliminated in fifteen ways. The result immediately required is obtained by eliminating ρ_1 and ρ_3, namely

$$[r_2r_3]R_1|a_1, A_1, a_3| - [r_1r_3]\rho_2|a_1, a_2, a_3| - [r_1r_3]R_2|a_1, A_2, a_3| + [r_1r_2]R_3|a_1, A_3, a_3| = 0$$

where the determinants are indicated by their first lines, from which the second and third lines are to be obtained by changing the letters without changing the suffixes, e.g.

$$|a_1, A_1, a_3| = \begin{vmatrix} a_1 & A_1 & a_3 \\ b_1 & B_1 & b_3 \\ c_1 & C_1 & c_3 \end{vmatrix}.$$

We have now to notice that these determinants are proportional to the perpendiculars to the plane

$$\begin{vmatrix} a_1 & x & a_3 \\ b_1 & y & b_3 \\ c_1 & z & c_3 \end{vmatrix} = 0$$

or the plane passing through the points (a_1, b_1, c_1), (a_3, b_3, c_3) and the origin, from the points (A_1, B_1, C_1), (a_2, b_2, c_2), (A_2, B_2, C_2) and (A_3, B_3, C_3); and these are the representative points of the directions of R_1, ρ_2, R_2, R_3 on the sphere of unit radius. The perpendiculars to the plane are therefore the sines of the perpendicular arcs to the great circle through (a_1, b_1, c_1), (a_3, b_3, c_3) and if these arcs are B_1', β_2', B_2', B_3' respectively (due regard being paid to sign) our equation becomes

$$[r_1r_3]\rho_2 \sin\beta_2' = [r_2r_3]R_1 \sin B_1' - [r_1r_3]R_2 \sin B_2' + [r_1r_2]R_3 \sin B_3' \ldots (4)$$

83. The points on the sphere just named are E_1, E_2, E_3, representing the heliocentric directions of the Earth and lying on the ecliptic, and P_1, P_2, P_3, representing the geocentric directions of the planet. The great circle mentioned is P_1P_3. Let this circle intersect the ecliptic in longitude H_2 and at the inclination η_2. Then we have the same relation between any one of the perpendicular arcs and the longitude (reckoned from H_2) and latitude of the point from which it is drawn as exists between the latitude of a point and its

right ascension and declination, the obliquity of the ecliptic being replaced by η_2. That is to say,

$$\begin{aligned}
\sin \beta_2' &= \cos \eta_2 \sin \beta_2 - \sin \eta_2 \cos \beta_2 \sin (\lambda_2 - H_2)\\
\sin B_1' &= \qquad\qquad\quad - \sin \eta_2 \sin (L_1 - H_2)\\
\sin B_2' &= \qquad\qquad\quad - \sin \eta_2 \sin (L_2 - H_2)\\
\sin B_3' &= \qquad\qquad\quad - \sin \eta_2 \sin (L_3 - H_2)
\end{aligned}$$

and as regards the points P_1, P_3

$$\begin{aligned}
0 &= \cos \eta_2 \sin \beta_1 - \sin \eta_2 \cos \beta_1 \sin (\lambda_1 - H_2)\\
0 &= \cos \eta_2 \sin \beta_3 - \sin \eta_2 \cos \beta_3 \sin (\lambda_3 - H_2).
\end{aligned}$$

The latter give, by addition and subtraction,

$$\begin{aligned}
2 \tan \eta_2 \sin \{\tfrac{1}{2} (\lambda_1 + \lambda_2) - H_2\} &= \sin (\beta_1 + \beta_3)/\cos \beta_1 \cos \beta_3 \cos \tfrac{1}{2} (\lambda_3 - \lambda_1)\\
2 \tan \eta_2 \cos \{\tfrac{1}{2} (\lambda_1 + \lambda_2) - H_2\} &= \sin (\beta_3 - \beta_1)/\cos \beta_1 \cos \beta_3 \sin \tfrac{1}{2} (\lambda_3 - \lambda_1)
\end{aligned}$$

and determine η_2 and H_2. We now put

$$c_1 = - R_1 \sin B_1'/\sin \beta_2', \quad c_2 = - R_2 \sin B_2'/\sin \beta_2', \quad c_3 = - R_3 \sin B_3'/\sin \beta_2'$$

and

$$n_1 = [r_2 r_3]/[r_1 r_3], \quad n_3 = [r_1 r_2]/[r_1 r_3].$$

The equation (4) then takes the simple form

$$\rho_2 = - c_1 n_1 + c_2 - c_3 n_3.$$

Now this is a purely geometrical relation involving the intersections of any plane through the Sun with three lines drawn in given directions through the positions of the Earth. If we imagine the plane to move into coincidence with the ecliptic, c_1, c_2, c_3 remain unaltered while in the limit ρ_1, ρ_2, ρ_3 vanish and r_1, r_2, r_3 become coincident with R_1, R_2, R_3. Hence if we put

$$\begin{aligned}
N_1 &= [R_2 R_3]/[R_1 R_3] = R_2 \sin (L_3 - L_2)/R_1 \sin (L_3 - L_1)\\
N_3 &= [R_1 R_2]/[R_1 R_3] = R_2 \sin (L_2 - L_1)/R_3 \sin (L_3 - L_1)
\end{aligned}$$

the equation

$$0 = - c_1 N_1 + c_2 - c_3 N_3$$

must be an identity, and this can be verified. Hence by the elimination of c_2

$$\rho_2 = c_1 (N_1 - n_1) + c_3 (N_3 - n_3) \qquad (5)$$

which is the required equation for ρ_2.

84. Since β_2' is the perpendicular arc from P_2 to $P_1 P_3$ it is geometrically evident that if the observed arcs of the planet's orbit are of the first order of small quantities (and we assume them to be small) β_2' is a quantity of the second order. Hence the equation (4) shows that if we are to obtain a value of ρ_2 which is a real approximation and not merely illusory we must at the outset employ values of the ratios of the triangles which are correct to the

second order in the time intervals. Accordingly we use (41) of § 61 and neglect the terms of higher order than the second; that is to say,

$$n_1 = \frac{\tau_1}{\tau_2}\left\{1 + \frac{\mu}{6r_2^3}(\tau_2^2 - \tau_1^2)\right\} \quad\text{.......................(6)}$$

$$n_3 = \frac{\tau_3}{\tau_2}\left\{1 + \frac{\mu}{6r_2^3}(\tau_2^2 - \tau_3^2)\right\} \quad\text{.......................(7)}$$

where

$$\tau_1 = t_3 - t_2, \quad \tau_2 = t_3 - t_1, \quad \tau_3 = t_2 - t_1.$$

It is necessary to neglect the mass of the planet and put $\mu = k^2$: this can safely be done in calculating a preliminary orbit, for which the perturbations are entirely neglected. The equation (5) for ρ_2 therefore becomes

$$\begin{aligned}\rho_2 &= c_1\left(N_1 - \frac{\tau_1}{\tau_2}\right) + c_3\left(N_3 - \frac{\tau_3}{\tau_2}\right)\\ &\quad - \frac{k^2\tau_1\tau_3}{6r_2^3}\left\{c_1\left(1 + \frac{\tau_1}{\tau_2}\right) + c_3\left(1 + \frac{\tau_3}{\tau_2}\right)\right\}\\ &= k_0 - l_0/r_2^3 \quad\text{.......................(8)}\end{aligned}$$

where k_0, l_0 are completely determined quantities. But if δ_2 is the angle ($< 180°$) between ρ_2 and R_2 produced,

$$r_2^2 = R_2^2 + \rho_2^2 + 2R_2\rho_2\cos\delta_2 \quad\text{.......................(9)}$$

where

$$\cos\delta_2 = \cos P_2E_2 = \cos\beta_2\cos(\lambda_2 - L_2).$$

If now ρ_2 be eliminated from (8), which corresponds to the definite form of Lambert's theorem (§ 77), and (9), an equation of the eighth degree in r_2 results. The nature of the roots of this form of equation has already been discussed in § 74. But Gauss replaced the eliminant by a much simpler equation which is easily found. We have

$$\frac{r_2}{\sin\delta_2} = \frac{R_2}{\sin z} = \frac{\rho_2}{\sin(\delta_2 - z)} \quad\text{.......................(10)}$$

where z is the angle subtended by R_2 at the planet in its intermediate observed position. Hence by (8)

$$\frac{R_2\sin(\delta_2 - z)}{\sin z} = k_0 - \frac{l_0\sin^3 z}{R_2^3\sin^3\delta_2}$$

or

$$l_0\sin^4 z/R_2^3\sin^3\delta_2 = -R_2\sin(\delta_2 - z) + k_0\sin z$$

and therefore if we put

$$\begin{aligned}m_0\cos q &= k_0 + R_2\cos\delta_2\\ m_0\sin q &= R_2\sin\delta_2\\ mm_0 &= l_0/R_2^3\sin^3\delta_2\end{aligned}$$

where m_0 is given the same sign as l_0, we have the simple form

$$m\sin^4 z = \sin(z - q) \quad\text{.......................(11)}$$

and this is the equation of Gauss. This form of equation does not avoid the possibility of an ambiguity arising from two distinct roots, which is inherent in the problem. But when only one appropriate root exists, it is easily found by successive approximation. In the most common case, that of a minor planet observed near opposition, $z - q$ is small and a first approximate value is given by

$$z_1 = q + m \sin^4 q.$$

When z is found the corresponding first approximations to ρ_2 and r_2 are given by (10).

85. We have now to find the corresponding values of ρ_1 and ρ_3. For this purpose we return to the equations (1), (2) and (3), and eliminate ρ_3 and R_3. The result can be written down at once in the form

$$[r_2 r_3]\,\rho_1\,|a_1, a_3, A_3| + [r_2 r_3]\,R_1\,|A_1, a_3, A_3| = [r_1 r_3]\,\rho_2\,|a_2, a_3, A_3| + [r_1 r_3]\,R_2\,|A_2, a_3, A_3|$$

or

$$n_1 \rho_1\,|a_1, a_3, A_3| + n_1 R_1\,|A_1, a_3, A_3| = \rho_2\,|a_2, a_3, A_3| + R_2\,|A_2, a_3, A_3|$$

where the determinants as before are represented by their first lines, the other rows being obtained by change of letters without change of suffixes. Since the same form of equation must remain true, the directions of ρ_1, ρ_2, ρ_3 being preserved, when the plane of the orbit is made to coincide with the ecliptic, in which case $\rho_1 = \rho_2 = 0$ and n_1 becomes N_1, the equation

$$N_1 R_1\,|A_1, a_3, A_3| = R_2\,|A_2, a_3, A_3|$$

must be an identity. Hence

$$n_1 \rho_1\,|a_1, a_3, A_3| = \rho_2\,|a_2, a_3, A_3| + (N_1 - n_1)\,R_1\,|A_1, a_3, A_3|.$$

Now

$$|a_1, a_3, A_3| = \begin{vmatrix} \cos\beta_1 \cos\lambda_1 & \cos\beta_3 \cos\lambda_3 & \cos L_3 \\ \cos\beta_1 \sin\lambda_1 & \cos\beta_3 \sin\lambda_3 & \sin L_3 \\ \sin\beta_1 & \sin\beta_3 & 0 \end{vmatrix}$$

$$= \cos\beta_1 \cos\beta_3\,\{-\tan\beta_1 \sin(\lambda_3 - L_3) + \tan\beta_3 \sin(\lambda_1 - L_3)\}$$

the axis of z being drawn towards the pole of the ecliptic and the axis of x towards the First Point of Aries. Similarly

$$|a_2, a_3, A_3| = \cos\beta_2 \cos\beta_3\,\{-\tan\beta_2 \sin(\lambda_3 - L_3) + \tan\beta_3 \sin(\lambda_2 - L_3)\}$$

and

$$|A_1, a_3, A_3| = \sin\beta_3 \sin(L_1 - L_3).$$

Hence

$$n_1 \rho_1 \cos\beta_1 = M_1 \rho_2 \cos\beta_2 + (N_1 - n_1)\,M_1' \quad \ldots\ldots\ldots\ldots\ldots(12)$$

where

$$M_1 = \frac{\tan\beta_2 \sin(\lambda_3 - L_3) - \tan\beta_3 \sin(\lambda_2 - L_3)}{\tan\beta_1 \sin(\lambda_3 - L_3) - \tan\beta_3 \sin(\lambda_1 - L_3)}$$

$$M_1' = \frac{R_1 \tan\beta_3 \sin(L_3 - L_1)}{\tan\beta_1 \sin(\lambda_3 - L_3) - \tan\beta_3 \sin(\lambda_1 - L_3)}.$$

Similarly the result of eliminating ρ_1 and R_1 from the original equations is to give (interchanging the suffixes 1 and 3)

$$n_3\rho_3 \cos\beta_3 = M_3\rho_2 \cos\beta_2 + (N_3 - n_3) M_3' \quad \text{.............(13)}$$

where

$$M_3 = \frac{\tan\beta_2 \sin(\lambda_1 - L_1) - \tan\beta_1 \sin(\lambda_2 - L_1)}{\tan\beta_3 \sin(\lambda_1 - L_1) - \tan\beta_1 \sin(\lambda_3 - L_1)}$$

$$M_3' = \frac{R_3 \tan\beta_1 \sin(L_1 - L_3)}{\tan\beta_3 \sin(\lambda_1 - L_1) - \tan\beta_1 \sin(\lambda_3 - L_1)}.$$

The coefficients M_1, M_1', M_3, M_3' as well as N_1, N_3 are constants throughout the process of approximation, but n_1, n_3 must be taken at this stage from the approximate forms (6) and (7). Then (12) and (13) give values of ρ_1 and ρ_3 corresponding to the approximate value of ρ_2 already obtained.

86. The heliocentric distances, longitudes and latitudes of the planet are next deduced by the formulae

$$\left.\begin{aligned} r_i \cos b_i \cos(l_i - L_i) &= \rho_i \cos\beta_i \cos(\lambda_i - L_i) + R_i \\ r_i \cos b_i \sin(l_i - L_i) &= \rho_i \cos\beta_i \sin(\lambda_i - L_i) \\ r_i \sin b_i &= \rho_i \sin\beta_i \end{aligned}\right\} \quad \text{.........(14)}$$

($i = 1, 2, 3$), which are at once found by taking the axis of x successively along R_1, R_2 and R_3, the axis of z being always directed towards the pole of the ecliptic. But these coordinates give the position of the plane of the orbit, for

$$\tan i \sin(l_1 - \Omega) = \tan b_1$$

$$\tan i \sin(l_3 - \Omega) = \tan b_3$$

where i is the inclination and Ω the longitude of the node; or in a form more suitable for calculation

$$\left.\begin{aligned} 2\tan i \sin\{\tfrac{1}{2}(l_1 + l_3) - \Omega\} &= \sin(b_1 + b_3)/\cos b_1 \cos b_3 \cos\tfrac{1}{2}(l_3 - l_1) \\ 2\tan i \cos\{\tfrac{1}{2}(l_1 + l_3) - \Omega\} &= \sin(b_3 - b_1)/\cos b_1 \cos b_3 \sin\tfrac{1}{2}(l_3 - l_1) \end{aligned}\right\} \quad (15)$$

And now the three arguments of latitude u_j, giving the differences of the true anomalies, can be calculated, for

$$\tan u_j = \tan(l_j - \Omega) \sec i \quad \text{.......................(16)}$$

($j = 1, 2, 3$). In the case of a comet, it is the practice to take $u_j <$ or $> 180°$ according as the latitude is positive or negative; in the case of a planet, u_j is placed in the same quadrant as $l_j - \Omega$. If we calculate n_1, n_3 from

$$n_1 = \frac{r_2 \sin(u_3 - u_2)}{r_1 \sin(u_3 - u_1)}, \quad n_3 = \frac{r_2 \sin(u_2 - u_1)}{r_3 \sin(u_3 - u_1)}$$

we shall not obtain improved values of these ratios, because these equations have a purely geometrical basis and merely serve as a useful control on the accuracy of the calculation; the values already obtained should be reproduced.

87. We have now arrived at preliminary approximations to the values of the geocentric distances ρ_1, ρ_2, ρ_3, the heliocentric distances r_1, r_2, r_3 and the arguments of latitude u_1, u_2, u_3. From these quantities we might proceed to deduce a complete set of elements. But our results are not accurate for two reasons: (1) the effect of aberration has been ignored, and (2) the expressions (6) and (7) employed for n_1 and n_3 were of necessity only approximate. The effect of aberration may be stated thus. The light observed at time t left the source whose distance is ρ at the time $t - \Delta t$, where

$$\Delta t = 498^{s}{\cdot}5\ \rho/1 \text{ day} = [7{\cdot}76116]\,\rho$$

in days, $498^{s}{\cdot}5$ being the light-time for unit astronomical distance. Had the source moved in the interval Δt uniformly with the velocity of the observer at time t, its position at time t would be correctly inferred from the observation, without correction, since in that case there is no relative motion between the source and the observer. If now we correct the observation for stellar aberration according to the ordinary rule the observer's motion attributed to the source is eliminated and we have the direction of the observed body at time $t - \Delta t$ from the observer's position at time t. This is the most convenient procedure in the present case, because it enables us to retain the Earth's coordinates (R, L) at the times of observation t throughout the calculation and to make no subsequent change in the planet's observed coordinates (λ, β) supposing them to be corrected for stellar aberration at the outset. This avoids many changes which would otherwise be necessary in the calculation of subsidiary quantities. It only remains when approximate values of ρ become known to correct the time t by subtracting Δt in so far as these relate to actual positions in the orbit. In particular, the corresponding corrections must be applied to the time intervals τ_1, τ_2, τ_3.

88. A better approximation to the values of n_1, n_3 might now be made by using the formulae of Gibbs or those of § 62 and with these values the whole calculation might be repeated. But we proceed at once to introduce the accurate formulae for the ratio of the sector to the triangle, (25) and (26) of § 55 in the case of an elliptic orbit. The sectors are

$$\tfrac{1}{2}y_1\,[r_2r_3], \quad \tfrac{1}{2}y_2\,[r_1r_3], \quad \tfrac{1}{2}y_3\,[r_1r_2]$$

and are proportional to τ_1, τ_2, τ_3 (now corrected for aberration). Hence

$$n_1 = \frac{y_2}{y_1}\cdot\frac{\tau_1}{\tau_2}, \qquad n_3 = \frac{y_2}{y_3}\cdot\frac{\tau_3}{\tau_2} \quad \text{.........................(17)}$$

Here

$$\left.\begin{aligned} y_2^2 &= m_2^2/(l_2 + \sin^2 \tfrac{1}{2} g_2) \\ y_2^3 - y_2^2 &= m_2^2\,(2g_2 - \sin 2g_2)/\sin^3 g_2 \end{aligned}\right\} \quad \text{...................(18)}$$

by the formulae quoted, and in the present notation

$$1 + 2l_2 = (r_1 + r_3)/2\sqrt{r_1r_3} \cos \tfrac{1}{2}(u_3 - u_1), \qquad m_2^2 = k^2\tau_2^2/\{2\sqrt{r_1r_3} \cos \tfrac{1}{2}(u_3 - u_1)\}^3.$$

The corresponding equations for y_1, y_3 can be written down by a symmetrical interchange of suffixes. Various methods have been devised for the convenient solution of these equations, generally involving the use of special tables.

In the absence of such tables, and they are not necessary, we may proceed thus. Writing the cubic equation in the form

$$y^3 - y^2 - \tfrac{4}{3}m^2 Q(2g) = 0, \quad Q(2g) = 3(2g - \sin 2g)/4\sin^3 g$$

where $Q(2g)$ approaches the value 1 as g approaches the value 0, we compare it with the identity

$$(\lambda^3 - \lambda^{-3}) - 3(\lambda - \lambda^{-1}) - (\lambda - \lambda^{-1})^3 = 0.$$

Thus $y = c/(\lambda - \lambda^{-1})$ if

$$\frac{c^3}{\lambda^3 - \lambda^{-3}} = \frac{c^2}{3} = \frac{4m^2 Q}{3}$$

that is, if $c = 2m\sqrt{Q} = \frac{1}{3}(\lambda^3 - \lambda^{-3})$. Hence if $\lambda^3 = \cot\frac{1}{2}\beta$, $3m\sqrt{Q} = \cot\beta$ and if $\lambda = \cot\frac{1}{2}\gamma$, $y = m\sqrt{Q}\tan\gamma$. But from the other equation in y we have $\sin\frac{1}{2}g = \sqrt{l}\tan\delta$ if $y = m\cos\delta/\sqrt{l}$.

Accordingly we throw the equations in y into the following form:

$$\left.\begin{aligned} \cot\beta &= 3m\sqrt{Q} \\ \tan^3\tfrac{1}{2}\gamma &= \tan\tfrac{1}{2}\beta \\ \cos\delta &= \sqrt{(lQ)}\tan\gamma \\ \sin\tfrac{1}{2}g &= \sqrt{l}\,.\tan\delta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Then, calculating the function Q with an approximate value g' of g, the result of solving these equations in turn is to lead to a new and closer approximation g''. With this new value the process is repeated until no change is found between the initial and final values. The true value of g has then been arrived at, and finally (the value of δ being taken from the last repetition)

$$y = m\cos\delta/\sqrt{l}.$$

Since $2g$ is the difference between the eccentric anomalies, the first approximation to its value may be taken to be the difference between the true anomalies, that is, between the arguments of latitude. When $2g$ is small, as it usually is in the practical problem, the direct calculation of the function $Q(2g)$ is inaccurate (cf. § 34). But if we write

$$\log Q(2g) = \tfrac{24576}{7000}\log\sec\tfrac{1}{2}g - \tfrac{17496}{7000}\log\sec\tfrac{1}{3}g$$

the error committed is practically negligible when $2g < 90°$, and the direct calculation only presents a difficulty when $2g$ is much smaller than this limit. The verification of this approximate formula may be left as an exercise.

It is unnecessary to repeat the solution of (19) until the value of g is exactly reproduced. This point may be explained in general terms as it is of wide application. Suppose the equations to be solved are $y = p(x)$, $x = q(y)$, p and q being any functions. These correspond to two curves P and Q. Starting with the approximate value x_1 we find $y_1 = p(x_1)$ and hence (x_1, y_1)

the point P_1 on P. Next we find similarly (x_2, y_1) the point Q_1 on Q. This gives the new value x_2 of x and with this we find successively (x_2, y_2) the point P_2 on P and (x_3, y_2) the point Q_2 on Q. But if the successive values x_1, x_2, x_3 do not differ greatly, the chords P_1P_2, Q_1Q_2 lie close to the curves P and Q and their intersection nearly coincides with the intersection of the curves. In this way we find for the correction to the third value x_3

$$x - x_3 = (x_2 - x_3)^2/\{(x_2 - x_1) - (x_3 - x_2)\}.$$

In the above case two solutions of (19) with application of the correction just indicated will generally suffice for the accurate determination of g and y.

89. When the values of y_1, y_2, y_3 have been thus obtained we have new values of n_1 and n_2 by (17). The next step is to recalculate ρ_2 by (5) and ρ_1, ρ_3 by (12) and (13). Hence r_1, r_2, r_3 and l_1, l_2, l_3 by (14), new values of Ω and i by (15) and finally u_1, u_2, u_3 by (16). This brings us back once more to the equations (18) in y. If the result of solving them with the improved values introduced is to leave n_1 and n_3 practically unaltered, our object is attained. Otherwise it is necessary to repeat the above steps until a satisfactory agreement is reached.

When this stage has been arrived at the problem has been solved, and it only remains to calculate the other elements of the orbit, Ω and i having been obtained in the last approximation. The three equations

$$p = r_j\{1 + e\cos(u_j - \omega)\}, \qquad (j = 1, 2, 3)$$

are linear in p, $e\cos\omega$ and $e\sin\omega$. The symmetrical solution gives

$$p = r_1r_2r_3\,\Sigma\sin(u_3 - u_2)/\Sigma\, r_2r_3\sin(u_3 - u_2)$$
$$-e\cos\omega = \Sigma\, r_2r_3(\sin u_3 - \sin u_2)/\Sigma\, r_2r_3\sin(u_3 - u_2)$$
$$e\sin\omega = \Sigma\, r_2r_3(\cos u_3 - \cos u_2)/\Sigma\, r_2r_3\sin(u_3 - u_2)$$

whence $e = \sin\phi$, $\omega = \varpi - \Omega$ and $a = p\sec^2\phi$. This, however, is not the simplest solution. The areal velocity $h = k\sqrt{p}$ (§ 26) and hence

$$k\tau_2\sqrt{p} = [r_1r_3]\,y_2 = y_2 r_1r_3\sin(u_3 - u_1) \quad\text{..................(20)}$$

Thus, p being known, we have

$$\left.\begin{aligned} \frac{p}{r_1} + \frac{p}{r_3} - 2 &= 2e\cos\tfrac{1}{2}(u_1 + u_3 - 2\omega)\cos\tfrac{1}{2}(u_3 - u_1) \\ \frac{p}{r_1} - \frac{p}{r_3} &= 2e\sin\tfrac{1}{2}(u_1 + u_3 - 2\omega)\sin\tfrac{1}{2}(u_3 - u_1) \end{aligned}\right\} \quad\text{.........(21)}$$

which also give e and ω. Finally, if the mass is neglected, the mean motion is $n = k''/a^{3/2}$ and the mean longitude at the epoch t_0 is (§ 64)

$$\epsilon = \omega + \Omega + E_j - e''\sin E_j - n(t_j - t_0) \quad\text{..................(22)}$$

where

$$\tan\tfrac{1}{2}E_j = \sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}(u_j - \omega), \qquad (j = 1, 2 \text{ or } 3).$$

The times t_j are here corrected for aberration (§ 87).

CHAPTER IX

DETERMINATION OF PARABOLIC AND CIRCULAR ORBITS

90. The method explained in principle in the last chapter requires no assumption as to the eccentricity of the orbit. Its practical convenience is greatest, however, when the eccentricity is comparatively small. On the other hand the majority of comets move in orbits almost strictly parabolic. For these it is important to have approximate elements after the first observations have been secured, in order that an ephemeris may be calculated to guide observers as to the position of the object. For this purpose the method of Olbers (published in 1797), which depends on the assumption of a parabolic orbit, has continued in use to the present time. Although only five elements have in this case to be determined we still use three complete observations of the comet giving the longitude and latitude (λ_j, β_j) at the three times t_j. We again take (R_j, L_j) as the corresponding radius vector and longitude of the Earth and ρ_j the geocentric distance of the comet, so that as before

$$x_j = a_j\rho_j + A_j R_j, \quad y_j = b_j\rho_j + B_j R_j, \quad z_j = c_j\rho_j + C_j R_j.$$

Here (x_j, y_j, z_j) are the heliocentric coordinates of the comet, (a_j, b_j, c_j) the direction cosines of ρ_j and (A_j, B_j, C_j) the direction cosines of R_j. In the ecliptic system of axes adopted,

$$a_j = \cos\lambda_j \cos\beta_j, \quad b_j = \sin\lambda_j \cos\beta_j, \quad c_j = \sin\beta_j.$$

We shall express ρ_3 in terms of ρ_1 and for this purpose it is possible to eliminate ρ_2 and R_2 from (1), (2) and (3) in § 82. The same result may, however, be deduced from the condition that the orbit is plane in another way.

91. If S is the Sun, E_1, E_2, E_3 the three positions of the Earth, and C_1, C_2, C_3 the three positions of the comet, S, C_1, C_2, C_3 are coplanar. Hence

$$\frac{[r_1 r_2]}{[r_2 r_3]} = \frac{\text{tetrahedron } SE_2C_1C_2}{\text{tetrahedron } SE_2C_2C_3}$$

$$= \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ A_2R_2 & B_2R_2 & C_2R_2 & 1 \\ a_1\rho_1 + A_1R_1, & b_1\rho_1 + B_1R_1, & c_1\rho_1 + C_1R_1, & 1 \\ a_2\rho_2 + A_2R_2, & b_2\rho_2 + B_2R_2, & c_2\rho_2 + C_2R_2, & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ A_2R_2 & B_2R_2 & C_2R_2 & 1 \\ a_2\rho_2 + A_2R_2, & b_2\rho_2 + B_2R_2, & c_2\rho_2 + C_2R_2, & 1 \\ a_3\rho_3 + A_3R_3, & b_3\rho_3 + B_3R_3, & c_3\rho_3 + C_3R_3, & 1 \end{vmatrix}}$$

$$= \begin{vmatrix} A_2 & B_2 & C_2 \\ a_1\rho_1+A_1R_1, & b_1\rho_1+B_1R_1, & c_1\rho_1+C_1R_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \div \begin{vmatrix} A_2 & B_2 & C_2 \\ a_2 & b_2 & c_2 \\ a_3\rho_3+A_3R_3, & b_3\rho_3+B_3R_3, & c_3\rho_3+C_3R_3 \end{vmatrix}$$

or, representing determinants by single rows,

$$[r_1r_2]\{\rho_3 \,|\, a_3, A_2, a_2 \,| + R_3 \,|\, A_3, A_2, a_2 \,|\} + [r_2r_3]\{\rho_1 \,|\, a_1, A_2, a_2 \,| + R_1 \,|\, A_1, A_2, a_2 \,|\} = 0.$$

But if, leaving the directions of ρ_1, ρ_2, ρ_3 unaltered, we move the plane of the orbit into coincidence with the ecliptic, we see that in the limit

$$[R_1R_2]\, R_3 \,|\, A_3, A_2, a_2 \,| + [R_2R_3]\, R_1 \,|\, A_1, A_2, a_2 \,| = 0$$

must be an identity. Hence

$$\rho_3 = -\frac{[r_2r_3]}{[r_1r_2]} \cdot \frac{|\, a_1, A_2, a_2 \,|}{|\, a_3, A_2, a_2 \,|}\rho_1 + \left\{\frac{[R_2R_3]}{[R_1R_2]} - \frac{[r_2r_3]}{[r_1r_2]}\right\} \frac{|\, A_1, A_2, a_2 \,|}{|\, a_3, A_2, a_2 \,|} R_1$$
$$= M\rho_1 + m.$$

Now

$$\begin{vmatrix} a_1, & A_2, & a_2 \\ b_1, & B_2, & b_2 \\ c_1, & C_2, & c_2 \end{vmatrix} = \begin{vmatrix} \cos\lambda_1 \cos\beta_1, & \cos L_2, & \cos\lambda_2\cos\beta_2 \\ \sin\lambda_1\cos\beta_1, & \sin L_2, & \sin\lambda_2\cos\beta_2 \\ \sin\beta_1\ , & 0\ , & \sin\beta_2 \end{vmatrix}$$
$$= \sin\beta_1\cos\beta_2\sin(\lambda_2 - L_2) - \sin\beta_2\cos\beta_1\sin(\lambda_1 - L_2)$$

and the other determinants can be written down by simple substitutions. Thus

$$M = \frac{[r_2r_3]}{[r_1r_2]} \cdot \frac{\sin\beta_1\cos\beta_2\sin(\lambda_2 - L_2) - \sin\beta_2\cos\beta_1\sin(\lambda_1 - L_2)}{\sin\beta_2\cos\beta_3\sin(\lambda_3 - L_2) - \sin\beta_3\cos\beta_2\sin(\lambda_2 - L_2)} \quad \text{..........(1)}$$

and

$$m = R_1\left\{\frac{[R_2R_3]}{[R_1R_2]} - \frac{[r_2r_3]}{[r_1r_2]}\right\} \frac{\sin\beta_2\sin(L_1 - L_2)}{\sin\beta_2\cos\beta_3\sin(\lambda_3 - L_2) - \sin\beta_3\cos\beta_2\sin(\lambda_2 - L_2)}.$$

In the practical problem the time intervals are usually small and it is possible to substitute the ratio of the sectors for the ratio of the triangles, both for the comet and the Earth, so that

$$\frac{[r_2r_3]}{[r_1r_2]} = \frac{[R_2R_3]}{[R_1R_2]} = \frac{t_3 - t_2}{t_2 - t_1} \quad \text{..........................(2)}$$

Thus $m = 0$ and with sufficient accuracy we may write

$$\rho_3 = M\rho_1 \quad \text{..(3)}$$

where M has the value given by (1) and (2), unless the comet is near the Sun and describes large arcs in comparatively short intervals. The effects of parallax and aberration are entirely neglected.

92. The next step is to express r_1, r_3 and the chord c joining the extremities of these radii in terms of ρ_1. We have

$$r_1^2 = \Sigma\,(a_1\rho_1 + A_1R_1)^2 = \rho_1^2 + R_1^2 + 2\rho_1R_1\cos\beta_1\cos(\lambda_1 - L_1) \quad \text{.........(4)}$$
$$r_3^2 = \Sigma\,(Ma_3\rho_1 + A_3R_3)^2 = M^2\rho_1^2 + R_3^2 + 2M\rho_1R_3\cos\beta_3\cos(\lambda_3 - L_3) \quad \text{...(5)}$$

and

$$c^2 = \Sigma \{(Ma_3 - a_1)\rho_1 + (A_3R_3 - A_1R_1)\}^2$$
$$= h^2\rho_1^2 + g^2 + 2\rho_1 hg \cos\phi \quad\ldots\ldots(6)$$

where

$$h^2 = \Sigma (Ma_3 - a_1)^2 = M^2 + 1 - 2M\{\sin\beta_1 \sin\beta_3 + \cos\beta_1 \cos\beta_3 \cos(\lambda_3 - \lambda_1)\}$$
$$g^2 = \Sigma (A_3R_3 - A_1R_1)^2 = R_3^2 + R_1^2 - 2R_1R_3\cos(L_3 - L_1)$$
$$hg\cos\phi = R_3\{M\Sigma a_3A_3 - \Sigma a_1A_3\} - R_1\{M\Sigma a_3A_1 - \Sigma a_1A_1\}$$
$$= M\cos\beta_3\{R_3\cos(\lambda_3 - L_3) - R_1\cos(\lambda_3 - L_1)\}$$
$$- \cos\beta_1\{R_3\cos(\lambda_1 - L_3) - R_1\cos(\lambda_1 - L_1)\}.$$

If E_1C is drawn equal and parallel to E_3C_3 it is clear that $CC_3 = E_1E_3 = g$, $CC_1 = h\rho_1$, $C_1C_3 = c$ and $C_1CC_3 = 180° - \phi$.

But Euler's equation gives

$$6k(t_3 - t_1) = (r_1 + r_3 + c)^{\frac{3}{2}} - (r_1 + r_3 - c)^{\frac{3}{2}}$$

and this must be satisfied by the appropriate value of ρ_1 in (4), (5) and (6). This value must be found by a process of approximation and for a suitable starting point we may consider c small in comparison with $r_1 + r_3$, $r_1 = r_3$ and $R_1 = 1$. Then

$$6k(t_3 - t_1) = (r_1 + r_3)^{\frac{3}{2}} . 3c/(r_1 + r_3) = 3\sqrt{2} . c\sqrt{r_1}$$

or

$$2k^2(t_3 - t_1)^2/h^2 = (\rho_1^2 + 2\rho_1\cos\phi . g/h + g^2/h^2)\{\rho_1^2 + 2\rho_1\cos\beta_1\cos(\lambda_1 - L_1) + 1\}^{\frac{1}{2}}.$$

With approximate values of the numbers which occur in this equation it is easy to find by trial a value of ρ_1 which is correct at least to one decimal place. Then with this value of ρ_1 it is possible to calculate c in two ways: (i) directly by (6), (ii) through r_1, r_3 given by (4) and (5) and inserted in Euler's equation, which may be written (§ 52) in the form

$$3k(t_3 - t_1)/\sqrt{2}(r_1 + r_3)^{\frac{3}{2}} = \sin\Theta, \quad c = 2\sqrt{2}(r_1 + r_3)\sin\tfrac{1}{3}\Theta\sqrt{\cos\tfrac{2}{3}\Theta} \ldots(7)$$

or solved by special tables. Two values of c thus correspond to a hypothetical value of ρ_1, and the latter must be varied until the discrepancy between the former is made to disappear. A rule analogous to that given in § 88 leads quickly to the desired value of ρ_1. For if the values ρ_1', ρ_1'' lead successively to the differences $\Delta_1 c$, $\Delta_2 c$ in c, it is easy to see that the value of ρ_1 to be inferred is given by

$$\rho_1 = \rho_1'' + (\rho_1'' - \rho_1')\Delta_2 c/(\Delta_1 c - \Delta_2 c).$$

In ordinary cases the correct result is quickly obtained in this way.

93. When ρ_1 and $\rho_3 = M\rho_1$ have been obtained it only remains to determine the elements of the orbit. The formulae of § 86 are again appropriate, namely

$$r_j\cos b_j\cos(l_j - L_j) = \rho_j\cos\beta_j\cos(\lambda_j - L_j) + R_j$$
$$r_j\cos b_j\sin(l_j - L_j) = \rho_j\cos\beta_j\sin(\lambda_j - L_j)$$
$$r_j\sin b_j \qquad = \rho_j\sin\beta_j$$

($j = 1, 3$), for the heliocentric distances, longitudes and latitude of the comet. Here r_1, r_3 should reproduce the values finally arrived at in the course of determining ρ_1. Also

$$2 \tan i \sin \{\tfrac{1}{2}(l_1 + l_3) - \Omega\} = \sin(b_1 + b_3)/\cos b_1 \cos b_3 \cos \tfrac{1}{2}(l_3 - l_1) \quad \text{...(8)}$$

$$2 \tan i \cos \{\tfrac{1}{2}(l_1 + l_3) - \Omega\} = \sin(b_3 - b_1)/\cos b_1 \cos b_3 \sin \tfrac{1}{2}(l_3 - l_1) \quad \text{...(9)}$$

($0 < i < 90°$ if $l_3 > l_1$, $90° < i < 180°$ if $l_3 < l_1$) give Ω and i. The arguments of latitude are given by

$$\tan u_j = \tan(l_j - \Omega) \sec i$$

($j = 1, 3$), where in this case $0 < u_j < 180°$ if $b_j > 0$. By the equation of the parabola

$$\sqrt{q} = \sqrt{r_1} \cos \tfrac{1}{2}(u_1 - \omega) = \sqrt{r_3} \cos \tfrac{1}{2}(u_3 - \omega) \quad \text{............(10)}$$

whence

$$\frac{\sqrt{r_3} - \sqrt{r_1}}{\sqrt{r_3} + \sqrt{r_1}} = \frac{\sin \frac{1}{4}(u_1 + u_3 - 2\omega) \sin \frac{1}{4}(u_3 - u_1)}{\cos \frac{1}{4}(u_1 + u_3 - 2\omega) \cos \frac{1}{4}(u_3 - u_1)}$$

or

$$\tan \tfrac{1}{4}(u_1 + u_3 - 2\omega) = \frac{\sqrt{r_3} - \sqrt{r_1}}{\sqrt{r_3} + \sqrt{r_1}} \cot \tfrac{1}{4}(u_3 - u_1) \quad \text{............(11)}$$

which gives $\omega = \varpi - \Omega$ and also q, the perihelion distance. Finally, T being the time of perihelion passage, we have (§ 29)

$$T = t_j - q^{\frac{3}{2}} \{\tan \tfrac{1}{2}(u_j - \omega) + \tfrac{1}{3} \tan^3 \tfrac{1}{2}(u_j - \omega)\} \sqrt{2}/k \quad \text{.........(12)}$$

($j = 1, 3$). This completes the determination of the five elements.

94. It is to be noticed that while the first and third observations have been completely used, the second observation has only entered partially into the calculation. In fact the five elements have been determined from six given coordinates in a unique way because λ_2, β_2 have not been used independently but only in the form $\cot \beta_2 \sin(\lambda_2 - L_2)$ in the equation (1) for M. Consequently it cannot be expected that the elements will satisfy the second place exactly and the magnitude of the discordance is an immediate test of the derived orbit. The second place is therefore calculated by finding (§ 29) $w_2 = u_2 - \omega$ from (12) ($j = 2$), $r_2 = q \sec^2 \frac{1}{2} w_2$, and hence the coordinates of the comet by means of

$$\begin{aligned} \rho_2 \cos \beta_2 \cos(\lambda_2 - \Omega) &= r_2 \cos u_2 - R_2 \cos(L_2 - \Omega) \\ \rho_2 \cos \beta_2 \sin(\lambda_2 - \Omega) &= r_2 \sin u_2 \cos i - R_2 \sin(L_2 - \Omega) \\ \rho_2 \sin \beta_2 &= r_2 \sin u_2 \sin i. \end{aligned}$$

If the residuals are small the elements may be considered satisfactory. If the residuals appear large, on the other hand, there are several possible reasons for the fact. There may be an error in the calculation, there may be an error in the observations, or the assumption of a parabolic orbit may be unjustified. The evidence of further observations must be the final test. But without additional material it is possible to improve the orbit obtained

by reconsidering the quantities which were ignored in the course of finding the first elements. Parallax and aberration may be allowed for. In the place of (3) may now be written

$$\rho_3 = \rho_1 (M + m/\rho_1)$$

where M and m are given by (1) and the following equation. At this stage an approximate value of ρ_1 is known and $[r_2 r_3]/[r_1 r_2]$ can be calculated with greater accuracy than by means of (2), for example by the application of the formulae of Gibbs or by direct calculation of the areas, since the sides of the triangles and the included angles are now approximately known. Thus the approximate M in (3) can now be replaced by the improved value $M + m/\rho_1$ and the remainder of the work can be repeated from this point. There are, however, shorter practical methods of removing a discrepancy in the middle place, which serve the purpose well enough since a provisional orbit is in general all that is required.

95. The eccentricities of planetary orbits are in general small and hence a circular orbit may prove a useful approximation to the true path, just as a parabolic orbit is a useful preliminary step towards the orbit of a periodic comet. As the eccentricity vanishes and the position of perihelion ceases to have a meaning, the number of elements to be determined is reduced to four and two complete observations of position only are required. Thus if a minor planet has been found on two photographs of the sky and no other observations are immediately available, a search ephemeris based on a circular orbit may be a useful guide in examining other plates which may have been taken at the same or at other observatories.

To consider the problem in a general form let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) be the geocentric coordinates of the Sun at the times of observation t_1, t_2 and let (l_1, m_1, n_1), (l_2, m_2, n_2) be the direction cosines of the observed directions of the planet. The axes may be any fixed system with the Sun at the origin. The planet is observed to lie on the lines

$$(x + X_1)/l_1 = (y + Y_1)/m_1 = (z + Z_1)/n_1 = \rho_1$$
$$(x + X_2)/l_2 = (y + Y_2)/m_2 = (z + Z_2)/n_2 = \rho_2$$

ρ_1, ρ_2 being the geocentric distances. Hence, if a is the radius of the orbit,

$$\begin{aligned} a^2 &= (l_1\rho_1 - X_1)^2 + (m_1\rho_1 - Y_1)^2 + (n_1\rho_1 - Z_1)^2 \\ &= \rho_1^2 - 2\rho_1 (l_1 X_1 + m_1 Y_1 + n_1 Z_1) + X_1^2 + Y_1^2 + Z_1^2 \\ &= \rho_2^2 - 2\rho_2 (l_2 X_2 + m_2 Y_2 + n_2 Z_2) + X_2^2 + Y_2^2 + Z_2^2 \end{aligned}$$

and, if n is the mean motion and $t_2 - t_1 = \tau$,

$$\begin{aligned} a^2 \cos n\tau &= (l_1\rho_1 - X_1)(l_2\rho_2 - X_2) + (m_1\rho_1 - Y_1)(m_2\rho_2 - Y_2) + (n_1\rho_1 - Z_1)(n_2\rho_2 - Z_2) \\ &= \rho_1\rho_2 \cos\theta - \rho_1 (l_1 X_2 + m_1 Y_2 + n_1 Z_2) - \rho_2 (l_2 X_1 + m_2 Y_1 + n_2 Z_1) \\ &\qquad + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 \end{aligned}$$

where θ is the angle between the observed directions. Since θ is a small angle the equation

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

is unsuitable for its determination, but the proper modification depends on the choice of coordinates. Similarly n cannot be accurately determined from $\cos n\tau$.

If we now put

$$A_1 = l_1 X_1 + m_1 Y_1 + n_1 Z_1, \quad A_2 = l_2 X_2 + m_2 Y_2 + n_2 Z_2$$

$$B_1 = l_1 X_2 + m_1 Y_2 + n_1 Z_2, \quad B_2 = l_2 X_1 + m_2 Y_1 + n_2 Z_1$$

we have

$$\begin{aligned} a^2 &= \rho_1^2 - 2A_1\rho_1 + X_1^2 + Y_1^2 + Z_1^2 \\ &= \rho_2^2 - 2A_2\rho_2 + X_2^2 + Y_2^2 + Z_2^2 \end{aligned}$$

$$a^2 \cos n\tau = \rho_1\rho_2\cos\theta - B_1\rho_1 - B_2\rho_2 + X_1X_2 + Y_1Y_2 + Z_1Z_2.$$

Hence

$$\begin{aligned} 4a^2\sin^2\tfrac{1}{2}n\tau = {} & \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\theta - 2(A_1 - B_1)\rho_1 - 2(A_2 - B_2)\rho_2 \\ & + (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 \\ = {} & \cos^2\tfrac{1}{2}\theta\,\{\rho_2 - \rho_1 - \tfrac{1}{2}(A_2 - A_1 - B_2 + B_1)\sec^2\tfrac{1}{2}\theta\}^2 \\ & + \sin^2\tfrac{1}{2}\theta\,\{\rho_2 + \rho_1 - \tfrac{1}{2}(A_2 + A_1 - B_2 - B_1)\operatorname{cosec}^2\tfrac{1}{2}\theta\}^2 \\ & + (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 \\ & - \tfrac{1}{4}(A_2 - A_1 - B_2 + B_1)^2\sec^2\tfrac{1}{2}\theta - \tfrac{1}{4}(A_2 + A_1 - B_2 - B_1)^2\operatorname{cosec}^2\tfrac{1}{2}\theta. \end{aligned}$$

The equations, which must be solved by trial, can therefore be reduced to the form

$$\left.\begin{aligned} &\sin\psi_1 = M_1/a, \quad \rho_1 = a\cos\psi_1 + A_1 \\ &\sin\psi_2 = M_2/a, \quad \rho_2 = a\cos\psi_2 + A_2 \\ &4a^2\sin^2\tfrac{1}{2}n\tau = \cos^2\tfrac{1}{2}\theta\,(\rho_2 - \rho_1 - b_1)^2 + \sin^2\tfrac{1}{2}\theta\,(\rho_2 + \rho_1 - b_2)^2 + c \end{aligned}\right\}\ldots(13)$$

where (without the transformations appropriate to the coordinate system)

$$M_1^2 = X_1^2 + Y_1^2 + Z_1^2 - A_1^2, \quad M_2^2 = X_2^2 + Y_2^2 + Z_2^2 - A_2^2$$

$$b_1 = (A_2 - A_1 - B_2 + B_1)/2\cos^2\tfrac{1}{2}\theta$$

$$b_2 = (A_2 + A_1 - B_2 - B_1)/2\sin^2\tfrac{1}{2}\theta$$

$$\begin{aligned} c = {} & (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 \\ & - (A_2 - B_2 - A_1 + B_1)^2/4\cos^2\tfrac{1}{2}\theta - (A_2 - B_2 + A_1 - B_1)^2/4\sin^2\tfrac{1}{2}\theta. \end{aligned}$$

A trial value of a gives, by (13), ψ_1, ψ_2 and hence ρ_1, ρ_2; these lead to a value of n and the process is continued until values are obtained consistent with the relation $n^2a^3 = k^2$. In the case of a minor planet $\log a = 0{\cdot}4$ is indicated as the appropriate initial value. With the above formulae the calculation can be performed directly in equatorial coordinates, and little will be gained by introducing the ecliptic system. When a and n have been

found, ρ_1, ρ_2 are also known by (13) and hence the heliocentric coordinates of the planet

$$x_1 = l_1\rho_1 - X_1, \quad y_1 = m_1\rho_1 - Y_1, \quad z_1 = n_1\rho_1 - Z_1$$
$$x_2 = l_2\rho_2 - X_2, \quad y_2 = m_2\rho_2 - Y_2, \quad z_2 = n_2\rho_2 - Z_2.$$

96. Gauss has given a method for finding a circular orbit, based on ecliptic coordinates. Let (R_1, L_1), (R_2, L_2) be the heliocentric distances and longitudes of the Earth at the times t_1, t_2 and (λ_1, β_1), (λ_2, β_2) the corresponding observed longitudes and latitudes of the planet. If in the plane triangle SE_1P_1 the angle at P_1 is denoted by z_1 and the exterior angle at E_1 by δ_1, $P_1SE_1 = \delta_1 - z_1$ and

$$a \sin z_1 = R_1 \sin \delta_1 \quad \text{.........................(14)}$$

Similarly in the triangle SE_2P_2, with similar notation,

$$a \sin z_2 = R_2 \sin \delta_2 \quad \text{.........................(15)}$$

The directions of the sides of the two triangles are now represented on a sphere of unit radius, SE_1, SE_2 being represented by E_1, E_2 on the ecliptic, SP_1, SP_2 by two points P_1, P_2. If G_1, G_2 represent E_1P_1, E_2P_2, these points lie respectively on the great circles E_1P_1, E_2P_2 and the arcs E_1G_1, E_2G_2 are δ_1 and δ_2. Let the circles E_1G_1, E_2G_2 cut the ecliptic at the angles γ_1, γ_2. Then the projections of the radius through G_1 on the radius through E_1, the radius through the point on the ecliptic 90° in advance of E_1 and the radius through the pole of the ecliptic give

$$\begin{aligned} \cos\beta_1 \cos(\lambda_1 - L_1) &= \cos\delta_1 \\ \cos\beta_1 \sin(\lambda_1 - L_1) &= \sin\delta_1 \cos\gamma_1 \\ \sin\beta_1 \qquad\qquad &= \sin\delta_1 \sin\gamma_1 \end{aligned}$$

and similarly

$$\begin{aligned} \cos\beta_2 \cos(\lambda_2 - L_2) &= \cos\delta_2 \\ \cos\beta_2 \sin(\lambda_2 - L_2) &= \sin\delta_2 \cos\gamma_2 \\ \sin\beta_2 \qquad\qquad &= \sin\delta_2 \sin\gamma_2 \end{aligned}$$

whence δ_1, δ_2 and γ_1, γ_2. Let the circles E_1P_1, E_2P_2 meet in D at an angle η. If $DE_1 = \phi_1$ and $DE_2 = \phi_2$, the analogies of Delambre applied to the triangle DE_1E_2 in which the side E_1E_2 is $L_2 - L_1$ and the adjacent angles are γ_1, $\pi - \gamma_2$, give

$$\frac{\sin\left\{\frac{\pi}{4} \pm \left(\frac{\pi}{4} - \frac{\phi_1 \mp \phi_2}{2}\right)\right\}}{\sin\left\{\frac{\pi}{4} \pm \frac{\pi}{4} - \frac{L_2 - L_1}{2}\right\}} = \frac{\sin\left\{\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{\pi - \gamma_2 \pm \gamma_1}{2}\right)\right\}}{\cos\left\{\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{\eta}{2}\right)\right\}}$$

or more explicitly

$$\begin{aligned} \sin\tfrac{1}{2}\eta \sin\tfrac{1}{2}(\phi_1 + \phi_2) &= \sin\tfrac{1}{2}(L_2 - L_1) \sin\tfrac{1}{2}(\gamma_2 + \gamma_1) \\ \sin\tfrac{1}{2}\eta \cos\tfrac{1}{2}(\phi_1 + \phi_2) &= \cos\tfrac{1}{2}(L_2 - L_1) \sin\tfrac{1}{2}(\gamma_2 - \gamma_1) \\ \cos\tfrac{1}{2}\eta \sin\tfrac{1}{2}(\phi_1 - \phi_2) &= \sin\tfrac{1}{2}(L_2 - L_1) \cos\tfrac{1}{2}(\gamma_2 + \gamma_1) \\ \cos\tfrac{1}{2}\eta \cos\tfrac{1}{2}(\phi_1 - \phi_2) &= \cos\tfrac{1}{2}(L_2 - L_1) \cos\tfrac{1}{2}(\gamma_2 - \gamma_1) \end{aligned}$$

whence ϕ_1, ϕ_2 and η. But since the arc $E_1P_1 = \delta_1 - z_1$ and $DE_1 = \phi_1$, $DP_1 = \phi_1 - \delta_1 + z_1$ and $DP_2 = \phi_2 - \delta_2 + z_2$, while $P_1P_2 = n(t_2 - t_1)$, n being the mean motion. Hence

$$\cos n(t_2-t_1) = \cos(\phi_1-\delta_1+z_1)\cos(\phi_2-\delta_2+z_2)+\sin(\phi_1-\delta_1+z_1)\sin(\phi_2-\delta_2+z_2)\cos\eta$$

or better, since $n(t_2 - t_1)$ is a small angle,

$$\sin^2 \tfrac{1}{2} n(t_2 - t_1) = \cos^2 \tfrac{1}{2}\eta \sin^2 \tfrac{1}{2}(\chi_1 + z_2 - z_1) + \sin^2 \tfrac{1}{2}\eta \sin^2 \tfrac{1}{2}(\chi_2 + z_2 + z_1) \quad \text{...(16)}$$

where

$$\chi_1 = \phi_2 - \delta_2 - (\phi_1 - \delta_1), \quad \chi_2 = \phi_2 - \delta_2 + (\phi_1 - \delta_1).$$

The solution is conducted in the usual way. Since δ_1, δ_2 are known an assumed value of a gives z_1, z_2 by (14) and (15). Then χ_1, χ_2 and η being known, the value of n is deduced from (16), and the process is continued until values are found which satisfy the relation $n^2a^3 = k^2$. When this has been done, the values of z_1, z_2 have also been found, and hence the geocentric distances are given by

$$\rho_1 \sin z_1 = R_1 \sin(\delta_1 - z_1), \quad \rho_2 \sin z_2 = R_2 \sin(\delta_2 - z_2)$$

but these distances are not actually required. Since the arc E_1P_1 on the sphere is $\delta_1 - z_1$ and makes the angle γ_1 with the ecliptic, we have the heliocentric longitude and latitude of P_1 (as in the case of G_1) given by

$$\begin{aligned}
\cos b_1 \cos(l_1 - L_1) &= \cos(\delta_1 - z_1) \\
\cos b_1 \sin(l_1 - L_1) &= \sin(\delta_1 - z_1)\cos\gamma_1 \\
\sin b_1 &= \sin(\delta_1 - z_1)\sin\gamma_1
\end{aligned}$$

with similar formulae for (l_2, b_2) the heliocentric longitude and latitude of the planet in its second position.

97. If (l_1, b_1), (l_2, b_2) have been thus obtained the remaining elements are easily found. For by (15) of § 86 the node and inclination are given by

$$\begin{aligned}
2\tan i \sin\{\tfrac{1}{2}(l_1 + l_2) - \Omega\} &= \sin(b_1 + b_2)/\cos b_1 \cos b_2 \cos\tfrac{1}{2}(l_2 - l_1) \\
2\tan i \cos\{\tfrac{1}{2}(l_1 + l_2) - \Omega\} &= \sin(b_2 - b_1)/\cos b_1 \cos b_2 \sin\tfrac{1}{2}(l_2 - l_1)
\end{aligned}$$

and then the arguments of latitude by

$$\tan u_1 = \tan(l_1 - \Omega)\sec i, \quad \tan u_2 = \tan(l_2 - \Omega)\sec i$$

with the check $u_2 - u_1 = n(t_2 - t_1)$. As the fourth element the argument of latitude u_0 at a chosen epoch t_0 may be taken, and this is simply

$$u_0 = u_1 + n(t_0 - t_1) = u_2 + n(t_0 - t_2)$$

where t_1, t_2 may be antedated for planetary aberration.

If, on the other hand, the heliocentric coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) have been found as in § 95, and i' is the inclination of the orbit to the

plane $z=0$ and Ω' is reckoned in this plane from the axis of x towards the axis of y, the plane of the orbit is

$$x \sin \Omega' \sin i' - y \cos \Omega' \sin i' + z \cos i' = 0$$

and as this is satisfied by the two points on the orbit we have

$$\frac{\sin \Omega' \sin i'}{y_1 z_2 - y_2 z_1} = \frac{\cos \Omega' \sin i'}{x_1 z_2 - x_2 z_1} = \frac{\cos i'}{x_1 y_2 - x_2 y_1}.$$

The solution can then be completed as before, the arguments u being now reckoned in the plane of the orbit from the node in the plane $z=0$.

The meaning of the quantities b_1, b_2 and c in § 95 may be seen thus. Let an axis of z be taken perpendicular to ρ_1 and ρ_2, and an axis of x midway between the directions of ρ_1 and ρ_2, so that (l_1, m_1, n_1) become $(\cos \frac{1}{2}\theta, -\sin \frac{1}{2}\theta, 0)$, (l_2, m_2, n_2) become $(\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta, 0)$, and (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) become (X_1', Y_1', Z_1'), (X_2', Y_2', Z_2'). Then

$$b_1 = (X_2' - X_1') \sec \tfrac{1}{2}\theta$$
$$b_2 = (Y_2' - Y_1') \operatorname{cosec} \tfrac{1}{2}\theta$$
$$c = (Z_2' - Z_1')^2.$$

If the difficulties of reducing this apparently simple problem to a practical form of calculation are carefully considered, in view of the small quantities which occur, the merit of the method in § 96 will be better understood. The reader must realize that the general problem of determining orbits from observations close together in time is essentially a question of arithmetical technique, and not of any particular mathematical difficulty. This is well illustrated in the history of the problem, especially in the eighteenth century.

It is to be remarked that the problem of finding a circular orbit to satisfy the given observations cannot always be solved. That a solution is not necessarily to be expected with arbitrary data can be readily seen, though the equations, not being algebraic, are too complicated to make a general discussion of the conditions feasible. It is enough to say that cases have occurred in practice in which a circular approximation to the orbit has proved impossible. The number of minor planets already discovered is approaching a thousand, and the most frequent eccentricity is in the neighbourhood of 0·12.

CHAPTER X

ORBITS OF DOUBLE STARS

98. There exist in the sky pairs of stars the components of which are separated by no more than a few seconds of arc, and frequently by less than one second. So close are they that they can only be seen distinctly in powerful telescopes, if indeed they can be clearly resolved at all. Such pairs are so numerous that probability forbids the idea that the contiguity of the stars can be explained by chance distribution in space. They must be physically connected systems for the most part and it is to be expected that the relative motion of the stars will reveal the effect of mutual gravitation. That this is actually true was discovered by Sir W. Herschel.

The motion is referred to the brighter component as a fixed point. The relative motion of the fainter component takes place in an ellipse of which the principal star occupies the focus (§ 24), unless there are other bodies in the system, or there proves to be no physical connexion between the pair. The apparent orbit which is observed is the projection of the actual orbit on the tangent plane to the celestial sphere, to which the line of sight to the principal star is normal, and since the point of observation is very distant compared with the dimensions of the orbit the projection can be considered orthogonal. Hence the law of areas holds also in the apparent orbit, which is equally an ellipse. But in this orbit the brighter star does not occupy the focus: its position gives the means of determining the relative situation of the true orbit.

The observations give the polar coordinates, ρ, θ, of the companion, the principal star being at the origin. The distance ρ is expressed in seconds of arc and the linear scale remains unknown unless the parallax of the system has been determined. The position angle θ is reckoned from the North direction through 360° in the order N., E. or following, S., W. or preceding. The planes of the actual and apparent orbits intersect in a line called the line of nodes and passing through the principal star. The position angle of that node which lies between 0° and 180° will be designated by Ω. Thus if the line of nodes is taken as the axis of ξ,

$$\xi = \rho \cos(\theta - \Omega), \quad \eta = \rho \sin(\theta - \Omega).$$

On the other hand, in the plane of the actual orbit, the longitude of periastron λ is the angle measured from this node to periastron in the direction of orbital motion. Hence in this plane, if the line of nodes is taken as the axis of x,

$$x = r\cos(w+\lambda), \quad y = r\sin(w+\lambda)$$

where r is the radius vector and w the true anomaly of the companion. But if i is the inclination of the two planes to one another, $\xi = x$ and $\eta = y\cos i$, so that

$$\rho\cos(\theta - \Omega) = r\cos(w+\lambda)$$

$$\rho\sin(\theta - \Omega) = r\sin(w+\lambda)\cos i.$$

Here the limits contemplated for i are $0°$ and $180°$. If $0° < i < 90°$, θ and w increase together with the time and the motion is direct. If $90° < i < 180°$, θ decreases with the time and the motion is retrograde. This is a departure from the more usual convention according to which i is always less than $90°$. It is then necessary to state whether the motion is direct or retrograde, and in the latter case to reverse the sign of $\cos i$. Ordinary visual observations of double stars, however, must leave the position of the orbital plane in one respect ambiguous, since there is nothing to indicate whether the node as defined is the approaching or receding node. The two possible planes intersect in the line of nodes and are the images of one another in the tangent plane to the celestial sphere.

In addition to the three elements, Ω, λ, i, now defined, four other elements are required. These are a, the mean distance in the true orbit, expressed like ρ in seconds of arc; e, the eccentricity of the true orbit; T, the time of periastron passage; and P, the period (or $n = 2\pi/P$, the mean motion) expressed in years.

99. The measurement of double stars is difficult and the early measures were very rough indeed. As the accuracy of the observations is not high refined methods of treatment are seldom justified and graphical processes have been largely employed. The observed coordinates may be plotted on paper and the apparent ellipse drawn through the points as well as may be. Let C be the centre and S the position of the principal star. The problem consists in finding the orthogonal projection by which the actual orbit is projected into this ellipse and the focus F into the point S.

The direction of the line of nodes can be determined by the principles of projective geometry. Conjugate lines through the focus F form an orthogonal involution. They project into an overlapping involution of conjugate lines through S. Of this involution one pair is at right angles and as in this case a right angle projects into a right angle it is clear that the line of nodes is parallel to one of the pair. Let SA, SA'; SB, SB' be two pairs of conjugate lines through S. When the apparent ellipse has been drawn these can be

found by drawing tangents at the extremities of chords through S; or by inscribing quadrangles in the ellipse, for each of which S is a harmonic point. On CS as diameter describe a circle, centre K. Let A_1, A_1'; B_1, B_1' be the points in which the conjugate lines intersect this circle and let A_1A_1', B_1B_1' intersect in O. Corresponding points of the same involution on the circle are obtained by drawing chords through O, and if OK meets the circle in N, N', SN, SN' are the orthogonal pair of the involution pencil required. Let $CABNA'B'$ be a transversal of the pencil drawn parallel to SN' so that AA', BB' subtend obtuse angles at S. This is an involution range of which N, since it corresponds to the point at infinity, is the centre, so that $AN \cdot NA' = BN \cdot NB'$. On NS take the point F such that NF^2 is equal to this constant product. Then F is the intersection of circles on the diameters AA', BB' and AFA', BFB' are right angles. Hence if NF be rotated about

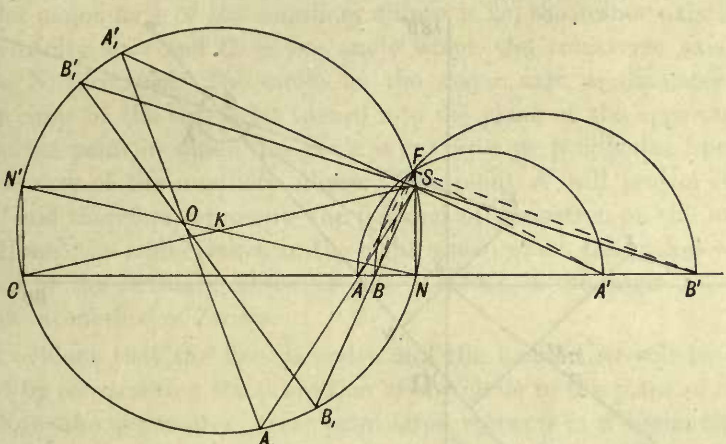


Fig. 4.

CN until FS is perpendicular to the plane CNS (the plane of the apparent orbit) right angles at F will be orthogonally projected into the involution of conjugate lines at S. The position of the focus F of the actual orbit has therefore been found, and the orthogonal projection by which the true and the apparent orbits are related.

The true orbit may be plotted point by point on the plane of the paper, with its centre C and focus F. For if P' is a point on the apparent orbit and P the corresponding point on the true orbit PP' is perpendicular to CN and PF, $P'S$ meet on CN. In particular, if X' (fig. 5) is a point where CS meets the apparent orbit, the corresponding point X in which the perpendicular through X' to CN meets CF is a vertex of the true orbit and $CX = a$. The eccentricity is given by

$$\frac{CS}{CX'} = \frac{CF}{CX} = e$$

and the inclination by

$$\frac{SN}{FN} = |\cos i|$$

where $0 < i < \frac{1}{2}\pi$ if the motion is direct and $\frac{1}{2}\pi < i < \pi$ if the motion is retrograde. Also Ω ($< \pi$) is the position angle of CN and λ is the angle between CN and CF measured in the direction in which the motion takes place. The five geometrical elements of the orbit have therefore been found.

100. It is to be noticed that this method does not require the ellipse which represents the apparent orbit to be actually drawn. When the observed positions have been plotted five points may be chosen to define the ellipse. These points need not be actual points of observation: it is better if they are graphically interpolated among the observed positions. Let them be denoted

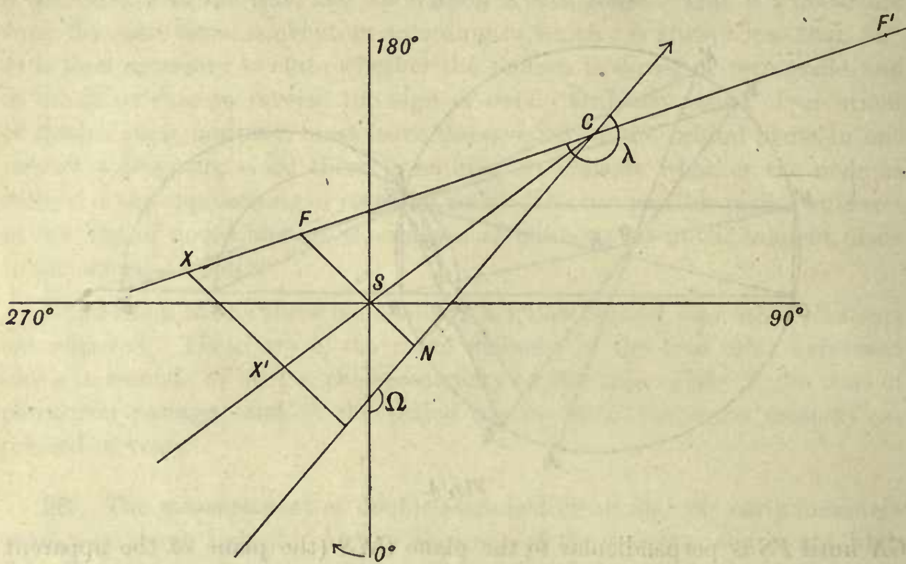


Fig. 5.

by 1, 2, 3, 4, 5. Draw a line through 1 parallel to 23. The second point in which this line meets the ellipse can then be found by Pascal's theorem with the ruler only. This gives two parallel chords and hence a diameter. Similarly a second diameter is drawn and the two intersect in the centre C of the apparent ellipse. Again, by a similar use of Pascal's theorem, the points in which the lines $1S$, $2S$, $3S$ meet the ellipse again are determined. This gives three pairs of lines each of which determines a quadrangle inscribed in the ellipse. If two of these be completed the sides of the harmonic triangles which meet in S determine two pairs of conjugate lines. From this point the construction follows as before. The point X' in which CS meets the apparent ellipse can be constructed by projective geometry. But it is unnecessary. If F' is the second focus of the real orbit and P the point

corresponding to any one of the assumed points on the apparent orbit, $FP + PF' = 2a$ and $CF = ae$. Hence a and e.

101. When the apparent ellipse has been drawn the eccentricity is known, for if CS meets the ellipse in X', the projection of the vertex of the true orbit, $CS/CX' = e$ since the ratio of segments of a line is unaltered by orthogonal projection. Let CY' be the conjugate diameter to CX' and therefore the projection of the minor axis of the true orbit. If the oblique ordinates parallel to CY' are produced in the ratio $1 : \sqrt{(1-e^2)}$ an auxiliary ellipse will be constructed which is clearly the projection of the auxiliary circle to the true orbit and has double contact with the apparent orbit, CS being the common chord. But the orthogonal projection of a circle is an ellipse of which the major axis is equal to the diameter and is parallel to the line of nodes, while the minor axis is the direct projection of the diameter. Hence the major axis of the auxiliary ellipse is $2a$, the minor axis $2a \cos i$, the eccentricity $\sin i$ and Ω is the angle which the transverse axis makes with the N. direction. The circle on the major axis as diameter is the auxiliary circle of the true orbit turned into the plane of the apparent orbit. Let X be the point in which this circle is cut by a perpendicular from X' to the major axis of the auxiliary ellipse. The point X will project into the point X' and therefore represents the position of periastron on the auxiliary circle. Hence the angle (taken in the right sense) which CX makes with the major axis of the auxiliary ellipse, or line of nodes, is the angle λ. This is the graphical method of Zwiers.

It is evident that the line of nodes and the inclination will be equally indicated by constructing the projection of any circle in the plane of the true orbit. Now the parameter p (or semi-latus rectum) is a harmonic mean between the segments of any focal chord. Hence the circle on the latus rectum as diameter has radii along any focal chord which are equal to the harmonic mean of the focal segments. The projection of this circle is an ellipse with its centre at S, its major axis equal to $2p$ and lying in the direction of the line of nodes, and its eccentricity equal to $\sin i$. This ellipse can be actually derived from the apparent orbit by laying off on radii through S lengths equal to the harmonic mean of the intercepts on the same chord between S and the curve, since the ratios are unaltered by projection. This principle, of which another use will be made, is due to Thiele.

102. Such graphical methods are tedious and may be avoided by a slight calculation when the apparent orbit has been drawn. Since the eccentricity is known when this has been done, there remain four geometrical elements, a, i, Ω, λ, to be determined. Four independent quantities are required and the four chosen by Sir John Herschel and others are 2α, the diameter through S, 2β the conjugate diameter, and χ_1, χ_2 the position angles of these diameters. The length of the chord through S parallel to β, or the projection of the latus

rectum of the true orbit, is therefore $2\beta\sqrt{(1-e^2)}$. Hence the relations between the positions in the true and apparent orbits (§ 98) give:

$$\begin{aligned}
\alpha(1-e)\ \cos(\chi_1-\Omega) &= a(1-e)\cos\lambda\\
\alpha(1-e)\ \sin(\chi_1-\Omega) &= a(1-e)\sin\lambda\cos i\\
\beta\sqrt{(1-e^2)}\cos(\chi_2-\Omega) &= -a(1-e^2)\sin\lambda\\
\beta\sqrt{(1-e^2)}\sin(\chi_2-\Omega) &= a(1-e^2)\cos\lambda\cos i
\end{aligned}$$

since $w=0°$ at periastron and $90°$ at the extremity of the latus rectum. Hence Ω is given by

$$\alpha^2(1-e^2)\sin 2(\chi_1-\Omega)+\beta^2\sin 2(\chi_2-\Omega)=0$$

or

$$\tan(\chi_1+\chi_2-2\Omega)=\tan(\chi_1-\chi_2)\cos 2\gamma$$

where

$$\tan\gamma=\sqrt{(1-e^2)}\,\alpha/\beta.$$

This equation in Ω is satisfied by $\Omega\pm\frac{1}{2}\pi$ as well as Ω. But

$$\cos^2 i=-\tan(\chi_1-\Omega)\tan(\chi_2-\Omega)$$

and this rejects $\Omega\pm\frac{1}{2}\pi$ since $\cos i<1$ and determines i. The first and third of the above set of four equations give both a and λ with its proper quadrant and the second or fourth gives also the proper sign of $\cos i$ (according to the convention of § 98). The solution is then free from ambiguity, understanding that χ_1 is the position angle corresponding to periastron and χ_2 the position angle when the companion has moved through one quadrant in its plane beyond this point.

103. Another method employs the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

of the apparent orbit referred to the principal star as origin. Without loss of generality c may be put equal to 1. The other coefficients are to be chosen to satisfy the observations as well as may be. But an elaborate solution is not justified because the one accurate element in the observation, the time, is not involved in this stage. The intersections of the ellipse with the axes and any fifth point give the result in the simplest way. The elements of the true orbit can then be derived in a variety of forms. Let us find the projection of the circle on the latus rectum. The above equation may be written

$$a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta+\frac{2}{\rho}(g\cos\theta+f\sin\theta)+\frac{c}{\rho^2}=0.$$

For a particular value of θ, ρ has two values, ρ_1 and $-\rho_2$, one positive and one negative since the origin is inside the curve. Hence, if ρ represents the harmonic mean,

$$\begin{aligned}
\frac{1}{\rho^2}=\frac{1}{4}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)^2&=\frac{1}{4}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2+\frac{1}{\rho_1\rho_2}\\
&=\{(g\cos\theta+f\sin\theta)^2-c(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)\}/c^2\\
&=(-B\cos^2\theta+2H\sin\theta\cos\theta-A\sin^2\theta)/c^2
\end{aligned}$$

where, in the usual notation,

$$A = bc - f^2, \quad H = fg - ch, \quad B = ac - g^2.$$

Hence the equation

$$Bx^2 - 2Hxy + Ay^2 + c^2 = 0$$

represents the projection of the circle on the latus rectum (§ 101), or an ellipse with axes $2p$ and $2p \cos i$ and its transverse axis coinciding with the line of nodes. It is therefore identical with the equation

$$\frac{(x \cos \Omega + y \sin \Omega)^2}{p^2} + \frac{(y \cos \Omega - x \sin \Omega)^2}{p^2 \cos^2 i} = 1$$

and thus

$$-B/c^2 = p^{-2} \cos^2 \Omega + p^{-2} \sec^2 i \sin^2 \Omega$$

$$H/c^2 = (p^{-2} - p^{-2} \sec^2 i) \sin \Omega \cos \Omega$$

$$-A/c^2 = p^{-2} \sin^2 \Omega + p^{-2} \sec^2 i \cos^2 \Omega$$

or

$$p^{-2} \tan^2 i \sin 2\Omega = -2H/c^2$$

$$p^{-2} \tan^2 i \cos 2\Omega = (B - A)/c^2$$

$$2p^{-2} + p^{-2} \tan^2 i = -(B + A)/c^2$$

which determine Ω, p and i.

Again, the perpendicular from the focus on the directrix is $a(e^{-1} - e) = pe^{-1}$. Hence the intercepts on the line of nodes and on the line perpendicular to it between the focus and the directrix are $p/e \cos \lambda$, $p/e \sin \lambda$. The projections of these intercepts, also at right angles, are $p/e \cos \lambda$, $p \cos i/e \sin \lambda$. But the projection of the directrix is the polar of the origin, or the line $gx + fy + c = 0$. Hence

$$(g \cos \Omega + f \sin \Omega)\, p/e \cos \lambda + c = 0$$

$$(-g \sin \Omega + f \cos \Omega)\, p \cos i/e \sin \lambda + c = 0$$

so that e and λ are given by the equations

$$e \sin \lambda = -p \cos i\,(f \cos \Omega - g \sin \Omega)/c$$

$$e \cos \lambda = \qquad -p\,(f \sin \Omega + g \cos \Omega)/c.$$

Equations for the five geometrical elements in the above form were first given by Kowalsky.

The form of the equation which represents the projection of a circle is defined by the fact that the asymptotes of the projected ellipse are parallel to the projection of the circular lines and therefore to the tangents from S to the apparent orbit. It will be found that the projection of the auxiliary circle, referred to its centre, is in the usual notation

$$C^2 (Bx^2 - 2Hxy + Ay^2) + \Delta^2 = 0$$

and that of the director circle

$$C^2(Bx^2 - 2Hxy + Ay^2) + \Delta(\Delta + Cc) = 0$$

while the eccentricity of the true orbit is given by

$$1 - e^2 = Cc/\Delta.$$

104. In some few cases a double star has been observed over more than one complete revolution. The period P is then known approximately and the date T of periastron passage, when the companion is situated on the diameter of the apparent orbit through S. Otherwise, when the geometrical elements have been determined, two dated observations suffice to determine these two additional elements. For two observed position angles θ_1, θ_2 give the corresponding true anomalies w_1, w_2 and hence the eccentric anomalies E_1, E_2, since

$$\tan(\theta - \Omega) = \tan(w + \lambda)\cos i, \quad \tan\tfrac{1}{2}E = \sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}w.$$

Then

$$n(t_1 - T) = E_1 - e\sin E_1, \quad n(t_2 - T) = E_2 - e\sin E_2$$

determine $n = 2\pi/P$ and T. In practice a larger number of such equations will be employed in order to reduce the effect of errors in the observations. The law of areas can also be applied directly to the apparent orbit, for if a_1 is the area described by the radius vector between the dates t_1, t_2, and A_1 is the area of the ellipse, $P = (t_2 - t_1) A_1/a_1$, and similarly T can be determined. A primitive method which has been used for measuring the areas consists in cutting out the areas in cardboard and weighing them.

When the parallax ϖ of a double star is known, a/ϖ is the mean distance in the system expressed in terms of the astronomical unit. Hence (§ 24), if m, m' are the masses of the components,

$$k^2(m + m') = 4\pi^2 a^3/\varpi^3 P^2$$

while $k^2 = 4\pi^2$ if the mass of the Sun-Earth system and the sidereal year are taken as units. For this purpose the mass of the Earth is negligible and thus, P being expressed in years,

$$m + m' = a^3/\varpi^3 P^2$$

is the combined mass of the system, compared with that of the Sun.

105. The apparent orbit can be reconstructed, on an arbitrary scale, from observed position angles alone. This course was advocated by Sir J. Herschel, who considered the measured distances of his day very inferior in accuracy. With this object the position angles are plotted as ordinates with the time as abscissa. Owing to inaccuracies the points will not lie exactly on a smooth curve, but such a curve must be drawn through them as well as possible. Let ψ be the angle which the tangent to the curve at the point

(t, θ) makes with the axis of t, so that $d\theta/dt = \tan\psi$. But since Kepler's law of areas is preserved in the apparent orbit, $\rho^2\dot{\theta} = h$, an undetermined constant. Hence $\rho = \sqrt{(h \cot \psi)}$ and the apparent orbit can thus be derived graphically from the (t, θ) curve. The elements with the exception of a can then be obtained and finally a is determined by the measured distances, of which no other use is made in the calculation.

The opposite case may arise, and is illustrated by the star 42 Comae Berenices, in which the determination of the elements must be based on the distances. Here the plane of the orbit passes through the point of observation, $i = 90°$ (or practically so) and the position angles serve only to determine Ω. If the star has been observed over more than one revolution the period P may be considered known. Corresponding to the point $(a \cos E, b \sin E)$ on the orbit, the observed distance is

$$\begin{aligned}\rho &= a \cos E \cos \lambda - b \sin E \sin \lambda - ae \cos \lambda \\ &= R \cos (E + \beta) - ae \cos \lambda\end{aligned}$$

while

$$n(t - T) = E - e \sin E.$$

If the observations are plotted for a single period, from maximum to maximum, the result is to give the curve

$$\begin{aligned}x &= nt = nT + E - e \sin E \\ y &= \rho = R \cos (E + \beta) - ae \cos \lambda\end{aligned}$$

which is a distorted cosine curve. Maximum and minimum correspond to $E = -\beta, \pi - \beta$ and give

$$\begin{aligned}nt_1 &= nT - \beta + e \sin \beta, & y_1 &= R - ae \cos \lambda \\ nt_2 &= nT + \pi - \beta - e \sin \beta, & y_2 &= -R - ae \cos \lambda\end{aligned}$$

whence R and $ae \cos \lambda$, while in addition

$$n(t_2 - t_1) = \pi - 2e \sin \beta.$$

These equations may be supplemented by a simple device. Taking the origin of x at the first maximum let the curve

$$y = R \cos x - ae \cos \lambda$$

also be drawn. Let P be a point on this curve and Q the corresponding point on the first curve such that the ordinates at P and Q are equal. Then at P, $x = E + \beta$, so that

$$QP = E + \beta - n(t - t_1) = e \sin E + \beta - n(T - t_1).$$

Hence the curve

$$y = e \sin (x - \beta) + \beta - n(T - t_1)$$

can be constructed by laying off on each ordinate through P a length equal to QP. This is a simple sine curve, the form of which will serve to show

any irregularities in the (nt, ρ) curve from which it is derived. The amplitude is $2e$, represented on the scale by which 2π corresponds to the period in x. The value of e being thus known gives β from $(t_2 - t_1)$ and hence a and λ, since

$$a \cos \lambda = R \cos \beta, \quad a \sin \lambda = R \sin \beta / \sqrt{(1 - e^2)}.$$

T is then given by the maximum and minimum of the original curve. But the sine curve has its maximum at $x = \beta + \frac{1}{2}\pi$ and its central line is $y = \beta - n(T - t_1)$. These conditions must also be fairly satisfied by the adopted solution.

106. Graphical methods, such as those sketched above, only provide a first approximation to the solution of a problem. Here in general the observations are too rough to make a closer approximation feasible. But if it is necessary to improve the elements thus found, each observation gives one equation in the following way. Let da, $d\Omega$, ... be the required corrections to the approximate elements, a, Ω, ... For the time t of an observation θ (or ρ) can be calculated. Its value is

$$\theta_c = f(t, a, \Omega, \ldots).$$

But the observed value is

$$\theta_o = f(t, a + da, \Omega + d\Omega, \ldots).$$

If then the elements have been found with such an accuracy that squares, products and higher powers of da, $d\Omega$, ... can be neglected,

$$\theta_o - \theta_c = \frac{\partial \theta}{\partial a} \cdot da + \frac{\partial \theta}{\partial \Omega} \cdot d\Omega + \ldots$$

a linear equation in da, $d\Omega$, ... And similarly with ρ. The coefficients are

$$\frac{\partial \theta}{\partial a} = 0, \qquad \frac{\partial \rho}{\partial a} = \frac{\rho}{a}$$

$$\frac{\partial \theta}{\partial \Omega} = 1, \qquad \frac{\partial \rho}{\partial \Omega} = 0$$

$$\frac{\partial \theta}{\partial i} = -\tfrac{1}{2} \sin 2(\theta - \Omega) \tan i, \qquad \frac{\partial \rho}{\partial i} = -\rho \sin^2(\theta - \Omega) \tan i$$

$$\frac{\partial \theta}{\partial \lambda} = \frac{r^2}{\rho^2} \cos i, \qquad \frac{\partial \rho}{\partial \lambda} = -\tfrac{1}{2}\rho \sin 2(\theta - \Omega) \sin i \tan i$$

$$\frac{\partial \theta}{\partial T} = -\frac{na^2}{\rho^2} \cos i \sqrt{(1 - e^2)}, \qquad \frac{\partial \rho}{\partial T} = -\frac{na^2}{r^2} \left\{ e\rho \sin E + \sqrt{(1 - e^2)} \frac{\partial \rho}{\partial \lambda} \right\}$$

$$\frac{\partial \theta}{\partial n} = -\frac{t - T}{n} \cdot \frac{\partial \theta}{\partial T}, \qquad \frac{\partial \rho}{\partial n} = -\frac{t - T}{n} \cdot \frac{\partial \rho}{\partial T}$$

$$\frac{\partial \theta}{\partial e} = \frac{r^2}{\rho^2} \left(\frac{a}{r} + \frac{1}{1 - e^2} \right) \sin w \cos i, \qquad \frac{\partial \rho}{\partial e} = \frac{\partial \rho}{\partial \lambda} \left(\frac{a}{r} + \frac{1}{1 - e^2} \right) \sin w - \frac{a\rho}{r} \cos w$$

the verification of which may be left as an exercise.

107. In some cases the position of a binary system has been measured relatively to some neighbouring star C which is independent of the system. Let A be the principal star, m_1 its mass, (x_1, y_1) its coordinates at the time t; and similarly let B be the companion, m_2 its mass, (x_2, y_2) its coordinates. A series of measures of AB gives

$$x_2 - x_1 = \rho \cos \theta, \quad y_2 - y_1 = \rho \sin \theta$$

while the measures of AC give $x_3 - x_1$, $y_3 - y_1$, (x_3, y_3) being the position of C. Let (ξ, η) be the C.G. of AB, so that

$$(m_1 + m_2)\,\xi = m_1 x_1 + m_2 x_2, \quad (m_1 + m_2)\,\eta = m_1 y_1 + m_2 y_2.$$

But the motions of C and of the C.G. of AB are uniform and independent. Hence

$$\xi = x_3 + \alpha + \beta t, \quad \eta = y_3 + \alpha' + \beta' t$$

where β, β' are the proper motions of the C.G. relative to C, and (α, α') is its position relative to C at the chosen epoch to which t refers. Thus

$$(m_1 + m_2)(x_3 + \alpha + \beta t) = m_1 x_1 + m_2 x_2$$

or

$$\alpha + \beta t - f(x_2 - x_1) + x_3 - x_1 = 0$$

and

$$\alpha' + \beta' t - f(y_2 - y_1) + y_3 - y_1 = 0$$

similarly, where

$$f = m_2/(m_1 + m_2).$$

From a series of such equations α, α', β, β' and f can be determined and therefore the ratio of the masses of A and B. But if a is the mean distance, P the period and ϖ the parallax of the system AB,

$$m_1 + m_2 = a^3/\varpi^3 P^2$$

and the masses of the individual stars, expressed in terms of the Sun, become known.

108. In certain cases the absolute coordinates of stars apparently single have exhibited a variable proper motion. It is then assumed that the variation is periodic and due to orbital motion in conjunction with an undetected body. The motion to be investigated is relative to the C.G. of the system, which itself is supposed to move uniformly. In the plane of the orbit the coordinates are $a'(\cos E - e)$, $b' \sin E$, and therefore in the plane of projection, when referred to the line of nodes and the line at right angles, they become

$$x = a'(\cos E - e) \cos \lambda - b' \sin E \sin \lambda$$
$$y = \{a'(\cos E - e) \sin \lambda + b' \sin E \cos \lambda\} \cos i.$$

Hence the orbital displacement in the direction of the position angle Q is

$$\xi = x \cos (\Omega - Q) - y \sin (\Omega - Q)$$
$$= g \cos E + h \sin E - ge$$

where

$$g = \quad a' \{\cos\lambda \cos(\Omega - Q) - \sin\lambda \sin(\Omega - Q) \cos i\}$$

$$h = - b' \{\sin\lambda \cos(\Omega - Q) + \cos\lambda \sin(\Omega - Q) \cos i\}$$

and $Q = 90°$ for displacements in R.A., $Q = 0°$ for displacements in declination. The observations of one coordinate, say δ, therefore give a series of equations of the form

$$\delta = \delta_0 + \mu_\delta t + g \cos E + h \sin E - ge$$

with

$$E - e \sin E = n(t - T).$$

From these e, n (or P), T, μ_δ, δ_0, g and h can be determined. Since g and h are functions of a', Ω, λ and i, these four elements cannot be derived from observations of one coordinate alone. But from observations of the other coordinate, say α, the corresponding quantities g' and h' can be found and the elements of the motion are then completely determinate, including a', the mean distance from the C.G. of the system.

In the two notable examples of this kind, Sirius and Procyon, the companion was discovered afterwards. It thus became possible to find the relative mean distance a and hence the ratio of the masses, since

$$m_1 a' = m_2 (a - a').$$

Hence, the parallax being known, the individual masses of the components have been determined. It is to be noticed that, when the companion cannot be observed, the function of the masses which can be found is $m_2^3 (m_1 + m_2)^{-2}$. For this is equal to $a'^3/\varpi^3 P^2$.

CHAPTER XI

ORBITS OF SPECTROSCOPIC BINARIES

109. Another class of orbits which are based on pure elliptic motion is presented by those systems which are known as spectroscopic binaries. It is now possible to determine the radial velocities of the stars in absolute measure with high accuracy. This follows from the application of Doppler's principle to the interpretation of stellar spectra. On the simple wave theory of light this principle is easily explained. A light disturbance travels outwards from its source in a spherical wave front which expands in the free ether of space with the uniform velocity U. Let a fixed set of rectangular axes be taken in this space, and let (x_1, y_1, z_1) be the position of the source at the origin of time. Let (u_1, v_1, w_1) be the velocity components of the source, supposed to be in uniform motion, and t the time at which a light disturbance is emitted. Similarly let (x_2, y_2, z_2) be the position of the observer, also supposed to be moving uniformly, (u_2, v_2, w_2) the velocity components, and τ the time at which the specified disturbance reaches him. For simplicity the motions have been considered uniform, but obviously they are immaterial except as regards the source at the instant t and the observer at the instant τ. Let the corresponding positions be A, B respectively and let the distance $AB = R$. Then

$$R^2 = \Sigma\, \{x_2 + u_2\tau - (x_1 + u_1 t)\}^2$$

$$\frac{dR}{dt} = \Sigma\alpha \left(u_2 \frac{d\tau}{dt} - u_1\right) = V_2 \frac{d\tau}{dt} - V_1$$

where (α, β, γ) are the direction cosines of AB and V_1, V_2 are the projections of the velocities (u_1, v_1, w_1), (u_2, v_2, w_2) on this line. But since the wave reaches B from A in the time $(\tau - t)$,

$$R = U(\tau - t), \quad \frac{dR}{dt} = U\left(\frac{d\tau}{dt} - 1\right).$$

Hence

$$\frac{d\tau}{dt} = \frac{U - V_1}{U - V_2} = 1 + \frac{V_2 - V_1}{U} + \frac{V_2(V_2 - V_1)}{U(U - V_2)}.$$

Now $(V_2 - V_1)$ is the component of relative velocity of A and B, measured in the direction of *separation* of the two points. This is a definite quantity. But V_2 is a component of the observer's absolute motion in free ether, and this is unknown. Presumably it is small in comparison with U, and the last term can be rejected as a negligible effect of the second order. Or, on the theory of relativity, V_2 is not only unknown but unknowable, and the effect is completely compensated by a transformation of the ideal coordinates of space and time into another set which is the subject of observation. All this has its counterpart in the theory of aberration, with which it is intimately related. Whether the limitation is imposed by the imperfection of practical observations or by the ultimate nature of things, it is necessary to be content with the effect of the first order.

If the light emitted at A has the wave length λ, the frequency of a particular phase in the wave train at A is U/λ. But the number of waves emitted in a time dt is received at B in the time $d\tau$. If then the apparent wave length of the light received at B is λ' and the apparent frequency U/λ',

$$U\lambda^{-1}\,dt = U\lambda'^{-1}\,d\tau$$

and therefore

$$\frac{\lambda'}{\lambda} = \frac{d\tau}{dt} = 1 + \frac{V}{U}$$

where V is the relative radial velocity of A from B. Thus the application of Doppler's principle gives

$$V = U\,.\,\Delta\lambda/\lambda$$

where $\Delta\lambda$ is the increase of wave length (or displacement measured positively towards the red end of the spectrum) of a spectral line, of which the natural wave length in the star is supposed known. Further details on the practical methods of reduction would be out of place here, and this explanation must suffice. It is usual to express V in km./sec., and the velocity of light may be taken to be $U = 299860$ km./sec.

110. From the measured radial velocity must be deduced the radial velocity of the star relative to the Sun, or rather relative to the centre of gravity of the solar system. This requires the calculation of certain corrections, of which the most important are due to (1) the diurnal rotation of the observer, and (2) the annual elliptic motion of the Earth relative to the Sun. The effects of perturbations of the Earth and Sun are comparatively small.

An observer situated on the equator is carried by the Earth's rotation over 40,000 km. in a sidereal day. This means a velocity of 0·46 km./sec. Hence the velocity of an observer in latitude ϕ is 0·46 $\cos\phi$ km./sec. always directed towards the E. point. If θ is the angular distance of the star from this point at the time of observation, $\cos\theta = \cos\delta\cos(h + 90°)$, where δ is the

declination and h the W. hour angle of the star. Hence the additive correction corresponding to (1) is

$$v_d = +0{\cdot}46 \cos\phi \cos\theta = -0{\cdot}46 \cos\phi \cos\delta \sin h.$$

Again, the Earth's elliptic velocity is compounded (§ 26) of one constant velocity V_1 perpendicular to the radius vector and another eV_1 perpendicular to the major axis, e being the eccentricity of the orbit. These vectors are directed to points in the ecliptic of which the longitudes are $\Theta - 90°$ and $\Gamma - 90°$, where Θ is the longitude of the Sun and Γ the longitude of the solar perigee. Let (l, β) be the star's longitude and latitude. Hence the required correction for the Earth's orbital motion is

$$v_a = + V_1 \cos\beta \{\cos(l - \Theta + 90°) + e\cos(l - \Gamma + 90°)\}.$$

Now V_1 is precisely that vector on which the constant of stellar aberration depends, so that if k'' is this constant,

$$V_1 = k''U/206265'' = 29{\cdot}76 \text{ km./sec.}$$

when the standard value of k, $20''{\cdot}47$, is adopted with the value of U given above. Hence the correction for (2) is

$$v_a = + 29{\cdot}76 \cos\beta \{\sin(\Theta - l) + e\sin(\Gamma - l)\}.$$

It is evident that the process might be reversed and the value of k determined by observing the apparent radial motion of one or more stars at different times of year. This has been done at the Cape Observatory, with the result that the standard value of k was reproduced very exactly, an excellent test of the theory. Indeed this is probably the best available method of finding the constant of aberration: it will be noticed that the adopted value of U, being a factor of both V_1 and V, will scarcely affect the resulting value of k.

When the necessary corrections have been applied to the apparent radial velocity of a star, the star's radial velocity is obtained relative to the solar system. This is affected by the motion of the latter relative to the stellar system as a whole. Hence conversely when the radial velocities of a number of stars scattered over the sky are known, it becomes possible to deduce the motion of the solar system relative to the average of those stars in absolute measure. If, further, ϖ is the parallax of a star, and μ its total annual proper motion, its transverse velocity is μ/ϖ when expressed in astronomical units per year. Now with the solar parallax $8''{\cdot}80$ and the Earth's equatorial radius 6378·249 km., the astronomical unit (or Earth's mean distance from the Sun) is 149,500,000 km. Hence this unit of velocity is equivalent to 4·737 km./sec. and the star's transverse velocity is $4{\cdot}737\ \mu/\varpi$ km./sec. Thus the velocity of a star relative to the Sun can be completely determined in absolute measure. This concerns questions of stellar kinematics which are now entering the region of dynamics but lie outside our present scope.

111. Repeated determinations of the radial velocity of a star yield values which in the majority of cases are constant within the errors of observation. The motion of the star is apparently uniform. But in other cases, perhaps a third of all the brighter stars, changes are observed which prove to be regular and periodic. These are attributed plausibly to the motion of one component in a binary system. Such spectroscopic binaries differ from visual doubles only in the scale of their orbits, which prevents them from being resolved even in the most powerful telescopes, while their periods are to be reckoned in days instead of years or even centuries. It may appear that the spectrum of the second component should also be seen. When the components are fairly equal in brightness, as in β Aurigae, this is so; the lines of the spectrum are seen periodically doubled. But with other stars, and this is the more common type, the companion is relatively so faint that only one spectrum is shown: it is quite unnecessary to suppose that the companion is then an absolutely dark body. Even when both spectra are visible the secondary spectrum is often difficult to detect and usually difficult to measure. As a particularly interesting example Castor (α Geminorum) may be quoted. The telescope reveals this star as a visual double, and the spectroscope shows that both components are themselves binary systems. More complex systems can be inferred from spectroscopic measures alone. Thus Polaris, which appears in the telescope as a single star, has been shown to be a triple system, consisting of a close pair revolving round a more distant third body. Here the motion will be considered in the first instance of one component of a binary system about the common centre of gravity, and it will be seen how far the elements of an elliptic orbit can be deduced from the measured radial velocities, these being based on the comparison of the star's spectrum with that from a terrestrial source (usually the spark spectrum of iron or titanium).

112. Since the period is generally short, the observations extend over several revolutions and the period P is determined by obvious considerations with fair exactness. This being known, the observed velocities can be referred to a single period with arbitrary epoch and plotted as ordinates with the time as abscissa in a diagram called the radial velocity curve. Such a curve is illustrated in fig. *a*, while the relative orbit is shown in fig. *b*, corresponding points being indicated by the same letters. The focus of this orbit is G, the centre of gravity of the system. The line of nodes AGB, passing through A the receding node and B the approaching node, is the line drawn through G in the plane of the orbit at right angles to the line of sight. The points P_1, P_2 mark the position of periastron and apastron, and the angle from GA to GP_1, measured in the direction of motion, is the longitude of periastron, ω. The true anomaly at any point of the orbit being w, the longitude of this point from A is $u = \omega + w$. Let i $(0° < i < 90°)$ be the

inclination of the orbit, this being the angle between its plane and the plane which is normal to the line of sight, and let e be the eccentricity.

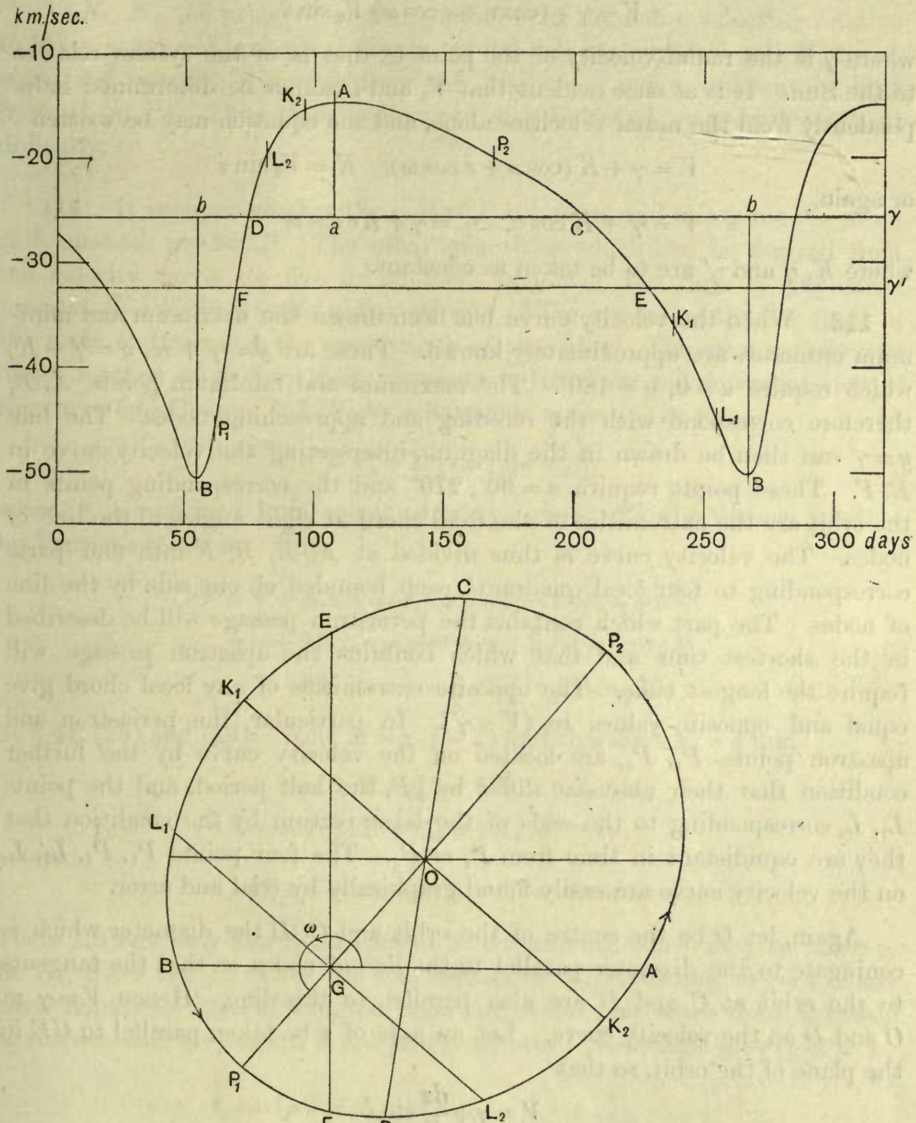


Fig. 6: (*a*) upper, (*b*) lower.

The orbital velocity of the star is compounded (§ 26) of one constant velocity V_2 transverse to the radius vector and another eV_2 perpendicular to the major axis. These may be resolved along and perpendicular to the line of nodes. The former components contribute nothing to the radial velocity. The latter are $+V_2 \cos u$ and $+eV_2 \cos \omega$ in the direction GE which is

drawn at right angles to GA. This line makes the angle $(90^\circ - i)$ with the line of sight, and hence the radial velocity which is measured is

$$V = \gamma + (\cos u + e \cos \omega)\, V_2 \sin i$$

where γ is the radial velocity of the point G, that is, of the system relative to the Sun. It is at once evident that V_2 and i cannot be determined independently from the radial velocities alone, and the equation may be written

$$V = \gamma + K(\cos u + e \cos \omega), \quad K = V_2 \sin i$$

or again,

$$V = \gamma' + K \cos u, \quad \gamma' = \gamma + Ke \cos \omega$$

where K, γ and γ' are to be taken as constant.

113. When the velocity curve has been drawn the maximum and minimum ordinates are approximately known. These are $y = \gamma' + K$, $y = \gamma' - K$, which require $u = 0$, $u = 180^\circ$. The maximum and minimum points, A, B, therefore correspond with the receding and approaching nodes. The line $y = \gamma'$ can then be drawn in the diagram, intersecting the velocity curve in E, F. These points require $u = 90^\circ$, 270° and the corresponding points in the orbit are the extremities of the focal chord at right angles to the line of nodes. The velocity curve is thus divided at A, E, B, F into four parts corresponding to four focal quadrants, each bounded on one side by the line of nodes. The part which contains the periastron passage will be described in the shortest time and that which contains the apastron passage will require the longest time. The opposite extremities of any focal chord give equal and opposite values to $(V - \gamma')$. In particular, the periastron and apastron points, P_1, P_2, are located on the velocity curve by the further condition that their abscissae differ by $\frac{1}{2}P$, the half period, and the points L_1, L_2 corresponding to the ends of the latus rectum by the condition that they are equidistant in time from P_1 or P_2. The four points P_1, P_2, L_1, L_2 on the velocity curve are easily found graphically by trial and error.

Again, let O be the centre of the orbit and COD the diameter which is conjugate to the diameter parallel to the line of nodes, so that the tangents to the orbit at C and D are also parallel to this line. Hence $V = \gamma$ at C and D on the velocity curve. Let an axis of z be taken parallel to GE in the plane of the orbit, so that

$$V = \gamma + \frac{dz}{dt} \sin i$$

$$\int_{t_1}^{t_2} (V - \gamma)\, dt = (z_2 - z_1) \sin i.$$

Now the integral represents the area of the velocity curve measured from the line $y = \gamma$. Hence by taking the limits at A, C, B, D it follows that the positive area of the velocity curve from A to C is equal to the negative area from C to B, and the negative area from B to D is equal to the positive area

from D to A. These conditions, which can be tested by a planimeter or some equivalent method, make it possible to draw the line $y=\gamma$ in the diagram.

At K_1, K_2, the extremities of the minor axis, the radial velocities relative to G are equal and opposite. Hence on the velocity curve K_1 and K_2 are at equal and opposite distances from the line $y=\gamma$ and equidistant in time from P_1 or P_2. Thus these points can also be found graphically without difficulty.

114. It is supposed that the period P is known, and this gives the mean daily motion, $\mu = 2\pi/P$. The other quantities which can be derived from the velocity curve are five in number, namely T the time of periastron passage, $K = V_2 \sin i$, γ the radial velocity of the system, ω the longitude of the node, and $e=\sin\phi$ the eccentricity of the orbit. The most satisfactory direct method of finding these elements is based on the representation of the curve (see Chapter XXIV) by a harmonic series in the form

$$V = V_0 + \Sigma r_j \sin(j\mu t + \beta_j)$$

where t is reckoned from some arbitrary epoch. This is always possible by Fourier's theorem. But

$$\begin{aligned} V &= \gamma + K\cos\omega\,(e+\cos w) - K\sin\omega\sin w \\ &= \gamma + 2K\cos\omega\cos^2\phi\,.\,e^{-1}\,\Sigma J_j(je)\cos jM \\ &\quad - 2K\sin\omega\cos\phi\,.\,\Sigma J_j'(je)\sin jM \end{aligned}$$

by § 41, (28) and (29). Now $M=\mu(t-T)$ and therefore $V_0=\gamma$ and

$$r_j\sin(j\mu T+\beta_j) = 2K_1\,.\,e^{-1}J_j\,(je)$$

$$-r_j\cos(j\mu T+\beta_j) = 2K_2\,.\,J_j'(je)$$

where

$$K_1 = K\cos\omega\cos^2\phi,\quad K_2 = K\sin\omega\cos\phi \quad\text{............(1)}$$

There are now only four quantities to be determined, which may be taken to be K_1, K_2, T and e. Thus the four equations corresponding to $j=1$, 2 are alone required: those of a higher order are useful only when there is reason to suspect that the motion is not purely elliptic. Now these give (§ 47)

$$\left.\begin{aligned} r_1\sin(\mu T+\beta_1) &= K_1\left(1-\frac{e^2}{8}+\frac{e^4}{192}-\dots\right) \\ -r_1\cos(\mu T+\beta_1) &= K_2\left(1-\frac{3e^2}{8}+\frac{5e^4}{192}-\dots\right) \end{aligned}\right\}\text{............(2)}$$

$$r_2\sin(2\mu T+\beta_2) = K_1 e\left(1-\frac{e^2}{3}+\frac{e^4}{24}-\dots\right)$$

$$-r_2\cos(2\mu T+\beta_2) = K_2 e\left(1-\frac{2e^2}{3}+\frac{e^4}{8}-\dots\right)$$

showing that r_2/r_1 is of the order of e. Hence, by division,

$$\frac{r_2}{r_1}\cdot\frac{\sin(2\mu T+\beta_2)}{\sin(\mu T+\beta_1)}=e\left(1-\frac{5e^2}{24}+\frac{e^4}{96}-\dots\right)$$

$$\frac{r_2}{r_1}\cdot\frac{\cos(2\mu T+\beta_2)}{\cos(\mu T+\beta_1)}=e\left(1-\frac{7e^2}{24}-\frac{e^4}{96}-\dots\right)$$

and, by subtraction and addition,

$$\frac{r_2}{r_1}\cdot\frac{\sin(\mu T+\beta_2-\beta_1)}{\sin 2(\mu T+\beta_1)}=\frac{e^3}{24}+\frac{e^5}{96}\dots$$

$$\frac{r_2}{r_1}\cdot\frac{\sin(3\mu T+\beta_2+\beta_1)}{\sin 2(\mu T+\beta_1)}=e-\frac{e^3}{4}\dots$$

the last equation containing no term in e^5. Eccentricities as high as 0·75 are met with occasionally, but even so it is evident that $(\mu T+\beta_2-\beta_1)$ is a very small angle which can scarcely exceed $2°$ and is generally negligible. If then

$$\alpha=\mu T+\beta_2-\beta_1$$

it is possible to neglect α^2 and the last equations become

$$\frac{r_2}{r_1}\cdot\alpha\operatorname{cosec}(4\beta_1-2\beta_2)=\frac{e^3}{24}+\frac{e^5}{96}\qquad\qquad(3)$$

$$\frac{r_2}{r_1}\cdot\{1+\alpha\cot(4\beta_1-2\beta_2)\}=e-\frac{e^3}{4}$$

whence

$$\frac{r_2}{r_1}+\left(\frac{e^3}{24}+\frac{e^5}{96}\right)\cos(4\beta_1-2\beta_2)=e-\frac{e^3}{4}.$$

From this equation e is easily found by trial and error, and then α, which gives T, is found from (3). The equations (2) give K_1 and K_2, whence finally K and ω are derived by (1). The process is therefore very simple, even without special tables, when once the harmonic representation of the velocity curve by two periodic terms has been obtained. This can be done very easily and with all needful accuracy by taking a sufficient number of equidistant ordinates from the curve.

115. It is, however, more usual in practice to find approximate preliminary elements by methods which are largely graphical and to improve them, if thought necessary, by a least-squares solution giving differential corrections. Thus $2K$ is the apparent range of the velocity curve, and when the periastron point P_1 has been located on the curve, T is known, while the areal property which fixes the position of the line $y=\gamma$ has been explained (§ 113). The remaining elements to be determined are therefore e and ω, and these are connected by the relation $Ke\cos\omega=\gamma'-\gamma$. A number of interesting properties have been used for the purpose.

Among these are the properties connected with a focal chord of the orbit. Let t_1 be the time at a certain point of the orbit and w and E_1 the

corresponding true and eccentric anomalies. Let t_2 be the time at the other end of the focal chord through the point and $180° + w$ and E_2 the true and eccentric anomalies. Then

$$(1-e)^{\frac{1}{2}} \tan \tfrac{1}{2}w = (1+e)^{\frac{1}{2}} \tan \tfrac{1}{2}E_1, \quad \mu(t_1 - T) = E_1 - e \sin E_1$$

$$-(1-e)^{\frac{1}{2}} \cot \tfrac{1}{2}w = (1+e)^{\frac{1}{2}} \tan \tfrac{1}{2}E_2, \quad \mu(t_2 - T) = E_2 - e \sin E_2.$$

Hence

$$-(1-e) = (1+e) \tan \tfrac{1}{2}E_1 \tan \tfrac{1}{2}E_2$$

or

$$e \cos \tfrac{1}{2}(E_2 + E_1) = \cos \tfrac{1}{2}(E_2 - E_1)$$

and therefore

$$\mu(t_2 - t_1) = E_2 - E_1 - 2e \sin \tfrac{1}{2}(E_2 - E_1) \cos \tfrac{1}{2}(E_2 + E_1)$$
$$= (E_2 - E_1) - \sin(E_2 - E_1).$$

Also

$$\tan \tfrac{1}{2}(E_2 - E_1) = -\tfrac{1}{2}(1-e^2)^{\frac{1}{2}} e^{-1} (\cot \tfrac{1}{2}w + \tan \tfrac{1}{2}w)$$
$$= -\cot \phi \operatorname{cosec} w.$$

Hence, if $2\eta = E_2 - E_1$,

$$\mu(t_2 - t_1) = 2\eta - \sin 2\eta, \quad \tan \phi \sin w = -\cot \eta.$$

Similarly, if t_3, t_4 are the times at the ends of the perpendicular chord, where the true anomalies are $90° + w$, $270° + w$,

$$\mu(t_4 - t_3) = 2\eta' - \sin 2\eta', \quad \tan \phi \cos w = -\cot \eta'.$$

The angles η, η' are easily found, especially with the help of a suitable table of the function $(x - \sin x)$, and hence ϕ or e and $w = u - \omega$. But the ordinate at the point t_1 gives $y - \gamma' = K \cos u$ and therefore u, whence the value of ω can be inferred. The equations

$$\tan \tfrac{1}{2}E_1 = \tan(45° - \tfrac{1}{2}\phi) \tan \tfrac{1}{2}w, \qquad \mu(t_1 - T) = E_1 - e \sin E_1$$

$$\tan \tfrac{1}{2}E_3 = \tan(45° - \tfrac{1}{2}\phi) \tan(\tfrac{1}{2}w + 45°), \quad \mu(t_3 - T) = E_3 - e \sin E_3$$

will give two independent values of T.

Sets of four points related in this way are easily located on the velocity curve, for they are given by $y - \gamma' = \pm K \cos u$, $\pm K \sin u$. Thus the four points $y - \gamma' = \pm K/\sqrt{2}$ are very suitable for the purpose. Here $u = 45°$, $w = 45° - \omega$. Two special sets have been mentioned in § 113, namely, AB, EF where $u = 0°$, $w = -\omega$, and P_1P_2, L_1L_2 where $w = 0°$. In the latter case $y - \gamma' = \pm K \cos \omega$, $\pm K \sin \omega$, giving ω immediately, $t_1 = T$, and e is given by $\phi = \eta' - 90°$.

116. There are also properties connected with a diameter of the orbit. If E is the eccentric anomaly at a point, $E + \frac{1}{2}\pi$ and $E + \frac{3}{2}\pi$ are the eccentric anomalies at the ends of the diameter conjugate to that which passes through the point. Let t_1, t_2 be the corresponding times. Then

$$\mu(t_1 - T) = E + \tfrac{1}{2}\pi - e \cos E$$
$$\mu(t_2 - T) = E + \tfrac{3}{2}\pi + e \cos E$$

so that

$$\tfrac{1}{2}\mu(t_1 + t_2 - 2T) = E + \pi$$

$$\tfrac{1}{2}\mu(t_2 - t_1 - \tfrac{1}{2}P) = e \cos E.$$

Now the points C, D, in which the line $y = \gamma$ cuts the velocity curve, satisfy this condition and the conjugate diameter being parallel to the line of nodes makes the angle $-\omega$ with the major axis. Hence in this case

$$-\tan\omega = \cos\phi . \tan E$$

and therefore

$$\begin{aligned}\tfrac{1}{2}\mu(t_2 - t_1 - \tfrac{1}{2}P) &= e(1 + \tan^2\omega \sec^2\phi)^{-\frac{1}{2}}\\ &= e\cos\omega(1 - e^2\cos^2\omega)^{-\frac{1}{2}}\cos\phi\end{aligned}$$

which gives $e = \sin\phi$ when $e\cos\omega = (\gamma' - \gamma)/K$ is known. Also

$$-e = \tfrac{1}{2}\mu(t_2 - t_1 - \tfrac{1}{2}P)\sec\tfrac{1}{2}\mu(t_1 + t_2 - 2T)$$

which gives a relation between e and T.

Another pair of such points is K_1, K_2, corresponding to the ends of the minor axis. Since $E = 0$ in this case,

$$\tfrac{1}{2}\mu(t_1 + t_2 - 2T) = \pi$$

$$\tfrac{1}{2}\mu(t_2 - t_1 - \tfrac{1}{2}P) = e.$$

Let u_1, u_2 be the longitudes at K_1, K_2. Then the radial velocities at these points, relative to G, are

$$\pm\tfrac{1}{2}K(\cos u_1 - \cos u_2) = \pm K\sin\tfrac{1}{2}(u_2 - u_1)\sin\tfrac{1}{2}(u_2 + u_1) = \pm K\cos\phi\sin\omega.$$

This quantity is therefore given by the ordinates at K_1, K_2 on the velocity curve, relative to the line $y = \gamma$.

117. The velocity curve also possesses interesting integral and differential properties which may be useful. It is necessary to have a consistent system of units, and since those of time and velocity have already been adopted, the unit of length is fixed and the natural system is:

Unit of time = 1 mean solar day = 86400 mean secs.,

Unit of length = 86400 km. = 0·0005779 astronomical units,

Unit of velocity = 1 km. per second,

Unit of mass = that of the Sun.

Now the constant of areal velocity in the orbit is

$$pV_2 = 2\pi ab/P = \mu a^2\cos\phi$$

so that

$$a\sin i = V_2\mu^{-1}\cos\phi\sin i = K\mu^{-1}\cos\phi.$$

The argument relative to the areas of the velocity curve in § 113 can now be made more precise. For the tangents to the orbit at C and D, referred to the principal axes of the ellipse, are

$$x\sin\omega + y\cos\omega = \pm\sqrt{(a^2\sin^2\omega + b^2\cos^2\omega)}$$

and the perpendiculars on them from the focus G are

$$z_1,\ z_2 = \pm ae \sin\omega + a\sqrt{(1 - e^2\cos^2\omega)}.$$

Measured from the line $y = \gamma$ let A_1 be the area of the velocity curve from A to C, $-A_1$ from C to B, $-A_2$ from B to D, and A_2 from D to A. Then

$$\tfrac{1}{2}(A_1 + A_2) = K\mu^{-1}\cos\phi\sqrt{(1 - e^2\cos^2\omega)}$$

$$\tfrac{1}{2}(A_1 - A_2) = K\mu^{-1}\cos\phi \,.\, e\sin\omega$$

$$A_1A_2 = K^2\mu^{-2}\cos^4\phi.$$

When A_1, A_2 have been measured in the proper units these equations determine ϕ (or e) and ω.

118. If the tangent to the velocity curve makes an angle ψ with the axis of time,

$$\tan\psi = \frac{dV}{dt} = -K\sin u\frac{dw}{dt}$$

and r being the radius vector in the orbit, the constant areal velocity is

$$\mu a^2\cos\phi = r^2\frac{dw}{dt}.$$

Hence

$$\tan\psi = -\mu K\cos\phi\sin u\,(a/r)^2$$

$$= -\mu K\sec^3\phi\sin u\,(1 + e\cos w)^2$$

and at special points on the curve $\tan\psi$ has these values:

A, B	$: u = 0^\circ, 180^\circ$	$: \tan\psi = 0$
E, F	$: u = 90^\circ, 270^\circ$	$: \tan\psi = \mp\mu K\sec^3\phi\,(1 \pm e\sin\omega)^2$
P_1, P_2	$: w = 0^\circ, 180^\circ$	$: \tan\psi = \mp\mu K\sec^3\phi\sin\omega\,(1 \pm e)^2$
L_1, L_2	$: w = 90^\circ, 270^\circ$	$: \tan\psi = \mp\mu K\sec^3\phi\cos\omega$
K_1, K_2	$: w = \pm(90^\circ + \phi)$	$: \tan\psi = \mp\mu K\cos\phi\cos(\omega \pm \phi).$

If $\tan\psi$ is found graphically at any of these points, attention must be paid to the scales in which ordinates and abscissae are represented. These expressions can then be used in order to find ω and ϕ.

Since

$$r \propto (\sin u\cot\psi)^{\frac{1}{2}},\quad w = u - \omega$$

and u at any point on the velocity curve is given by the ordinate measured from the axis $y = \gamma'$, it is possible theoretically to plot the actual orbit to an arbitrary scale, point by point. This is scarcely a practical method, but deserves mention as the counterpart of Sir John Herschel's method for double star orbits (§ 105).

119. The values of the elements found by any of these graphical methods are approximate only. They can be improved by the addition of differential corrections, δK to K, δe to e, $\delta\omega$ to ω, δT to T and $\delta\mu$ to μ. Thus each observation gives an equation of condition of the form

$$V_o - V_c = \delta\gamma' + \cos u \, . \, \delta K - K \sin u \, . \, \delta\omega - K \sin u \left(\frac{\partial w}{\partial e}\delta e + \frac{\partial w}{\partial T}\delta T + \frac{\partial w}{\partial \mu}\delta\mu\right)$$

and it is easily found that

$$\frac{\partial w}{\partial e} = \sin w \, (2 + e \cos w) \sec^2 \phi$$

$$\frac{\partial w}{\partial T} = -\mu \, (1 + e \cos w)^2 \sec^3 \phi$$

$$\frac{\partial w}{\partial \mu} = (t - T) \, (1 + e \cos w)^2 \sec^3 \phi.$$

It is more usual to give γ, the radial velocity of the system, than γ', but this quantity can be derived finally from the relation $\gamma = \gamma' - Ke \cos\omega$.

120. When the elements of an orbit specified above have been obtained, by whatever method, some information can be gained as to the dimensions and mass of the system. An equation already found in § 117 gives

$$a \sin i = K\mu^{-1} \cos\phi \, . \, 86400 \text{ km.}$$

when the unit of length there adopted is explicitly introduced. Let m be the mass of the star whose spectrum is observed, and m' the mass of the other star. Then

$$\mu^2 a^3 \left(1 + \frac{m}{m'}\right)^3 = (m + m') \, C$$

where C is a constant depending on the units employed. These being as stated in § 117, the special case when $m' = 1$, $m = 0$, gives

$$C = \frac{4\pi^2}{(365{\cdot}25)^2} \cdot \frac{1}{(0{\cdot}0005779)^3}, \quad \log C = 6{\cdot}18557.$$

It follows that

$$m'^3 (m + m')^{-2} \sin^3 i = [3{\cdot}81443 - 10] \, K^3\mu^{-1} \cos^3\phi$$
$$= [3{\cdot}01625 - 10] \, K^3 P \cos^3\phi$$

and it is only this function of the masses, involving the unknown inclination of the orbit, which can be determined when only one spectrum can be observed.

If, however, the radial velocity V' of the second component of the system can be measured at the same time, which is possible when the two superposed spectra are of comparable intensity,

$$m \, (V - \gamma) + m' \, (V' - \gamma) = 0.$$

One such equation will give the ratio $m : m'$ when γ is known and two will give γ in addition without any knowledge of the orbit. It has been supposed that the radial velocities have been determined by referring the stellar spectrum to a comparison spectrum from a terrestrial source, as mentioned in § 111. When there is no comparison spectrum, as when an objective prism is used, and the stellar spectrum shows double lines, it is still possible to deduce the orbit of the system from the relative displacements of corresponding lines. But the orbit is then the relative orbit, a is the mean distance of the components from one another, and it is easily seen that $(m + m') \sin^3 i$ must be substituted for the above function of the masses.

121. The true spectroscopic binary cannot be resolved in the telescope. But one or both components of a visual double can, when bright enough, be observed with the spectrograph, and very interesting results can be gained in this way. Let a, a' be the mean distances of the components relative to the centre of mass, expressed in terms of the linear unit 86400 km. The astronomical unit contains 1730 such units. Let a'' be the visual mean distance and ϖ'' the parallax of the system, both expressed in seconds of arc. Then

$$ma = m'a' = \frac{mm'}{m + m'}(a + a')$$

$$= 1730 \cdot \frac{a''}{\varpi''} \cdot \frac{mm'}{m + m'}$$

and therefore

$$V = \gamma + K(\cos u + e \cos \omega)$$

$$= \gamma + \mu a \sin i \sec \phi (\cos u + e \cos \omega)$$

$$= \gamma + 1730 \mu \sin i \sec \phi (\cos u + e \cos \omega) \cdot \frac{a''}{\varpi''} \cdot \frac{m'}{m + m'}$$

while for the other component similarly

$$V' = \gamma - 1730 \mu \sin i \sec \phi (\cos u + e \cos \omega) \cdot \frac{a''}{\varpi''} \cdot \frac{m}{m + m'}.$$

If then the elements of the visual orbit have been independently determined and the radial velocity of the first component alone can be observed at different dates, the two quantities γ and $(1 + m/m')\varpi''$ can be inferred. If the radial velocity of the second component can also be observed, the parallax, the ratio of the masses and hence the individual masses themselves in terms of the Sun (§ 104) can also be deduced. From the relative radial velocity alone,

$$V - V' = 1730 \mu \sin i \sec \phi (\cos u + e \cos \omega) a''/\varpi''$$

the parallax can be found, and hence the total mass of the system.

One question remains in the determination of the true orientation of a double star orbit in space, which can only be decided by radial velocity

observations. For the spectroscopic binary i has been defined so that $0 < i < \frac{1}{2}\pi$, while for the visual double $0 < i < \pi$. This difference does not affect $\sin i$, which is positive in either case. Hence if V_1, V_2 are the radial velocities of the principal star at different times, the two expressions

$$V_1 - V_2, \quad \cos(w_1 + \omega) - \cos(w_2 + \omega)$$

have the same sign, where ω is the longitude of periastron of this star, reckoned from its receding node in the direction of motion. But λ is the longitude of periastron of the companion at its first node Ω ($< \pi$). Hence if the expressions

$$V_1 - V_2, \quad \cos(w_1 + \lambda) - \cos(w_2 + \lambda)$$

have the same sign, $\lambda = \omega$. This means that the principal star is receding and the companion is approaching when the latter is at its node Ω. If on the other hand the expressions are of opposite signs, $\lambda = \omega + \pi$ and the companion is receding at Ω.

Otherwise it may be possible to determine the velocities V, V' of the principal star and the companion respectively at the same time. Then the expressions

$$V - V', \quad \cos(w + \omega) + e\cos\omega$$

have the same sign, and therefore if the expressions

$$V - V', \quad \cos(w + \lambda) + e\cos\lambda$$

have the same sign, $\lambda = \omega$, while if they have opposite signs, $\lambda = \omega + \pi$. The same consequences follow as before. Thus a knowledge of either $V_1 - V_2$ or $V - V'$ removes the ambiguity with regard to the true position of the orbital plane, which remains after the elements of a double star have been determined from visual observations alone.

CHAPTER XII

DYNAMICAL PRINCIPLES

122. It will be convenient in this chapter to recall some of the salient features of dynamical theory and to consider as briefly as possible the form of those transformations which are of the greatest importance in astronomical applications. We shall start from Lagrange's equations.

Let the system consist of a number of particles whose coordinates can be expressed in terms of n quantities $q_1, q_2, \ldots, q_n$ and possibly of the time t. Let m be the mass of a typical particle situated at the point (x, y, z). Then

$$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial q_1} \cdot \dot{q}_1 + \ldots + \frac{\partial x}{\partial q_n} \cdot \dot{q}_n$$

so that

$$\frac{\partial \dot{x}}{\partial \dot{q}_r} = \frac{\partial x}{\partial q_r}.$$

Hence

$$\frac{d}{dt}\left(\tfrac{1}{2} m \frac{\partial \dot{x}^2}{\partial \dot{q}_r}\right) = m \frac{d}{dt}\left(\dot{x} \frac{\partial x}{\partial q_r}\right)$$

$$= X \frac{\partial x}{\partial q_r} + m\dot{x} \frac{\partial \dot{x}}{\partial q_r}$$

where X is the component of the force acting on m. If T is the kinetic energy of the whole system,

$$T = \Sigma \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Hence adding all the equations of the preceding type for the three coordinates and all the particles,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \Sigma \left(X \frac{\partial x}{\partial q_r} + Y \frac{\partial y}{\partial q_r} + Z \frac{\partial z}{\partial q_r}\right) + \frac{\partial T}{\partial q_r}.$$

Now the forces which occur in astronomical problems are in general conservative, and we can write

$$\Sigma (X dx + Y dy + Z dz) = - dU$$

where dU is a perfect differential. U represents the work done by the forces in a change from the actual configuration to some standard configuration and is called the potential energy. We therefore have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \frac{\partial (T-U)}{\partial q_r}.$$

But U does not contain $\dot{q}_r$, and hence, if we write $T = U + L$, this becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \frac{\partial L}{\partial q_r}, \quad (r = 1, 2, \ldots, n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

which is the standard form of Lagrange's equations.

The function L is often called the Kinetic Potential. In the absence of moving constraints (or some analogous feature) within the system $\frac{\partial x}{\partial t} = \ldots = 0$. Then T is a homogeneous (positive definite) quadratic form in $\dot{q}_1, \ldots, \dot{q}_n$.

123. If L does not contain t explicitly, the equations admit an integral called the Integral of Energy. For in this case

$$\begin{aligned}\frac{dL}{dt} &= \sum_r \left(\frac{\partial L}{\partial q_r} \cdot \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r} \cdot \ddot{q}_r\right)\\ &= \sum_r \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) \cdot \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r} \cdot \ddot{q}_r\right\}\\ &= \frac{d}{dt}\left(\sum_r \frac{\partial L}{\partial \dot{q}_r} \cdot \dot{q}_r\right)\end{aligned}$$

so that

$$\sum_r \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where h is a constant of integration. Replacing L by $T - U$, where T is a homogeneous quadratic form in $\dot{q}_r$ and U does not contain $\dot{q}_r$, we have

$$h = 2T - (T - U) = T + U$$

which shows that h is the sum of the kinetic and potential energies.

More generally, let L contain t explicitly through U and let T no longer be a homogeneous function in $\dot{q}_r$ but of the form $T_2 + T_1 + T_0$, where T_2 is a homogeneous quadratic function, T_1 a linear function and T_0 of no dimensions in $\dot{q}_r$. Then similarly

$$\begin{aligned}\frac{dL}{dt} &= \frac{d}{dt}\left(\Sigma_r \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r\right) + \frac{\partial L}{\partial t}\\ &= \frac{d}{dt}\left(\Sigma_r \frac{\partial T}{\partial \dot{q}_r} \dot{q}_r\right) - \frac{\partial U}{\partial t}\\ &= \frac{d}{dt}(2T_2 + T_1) - \frac{\partial U}{\partial t}\end{aligned}$$

or since $L = T_2 + T_1 + T_0 - U$

$$\frac{d}{dt}(T_2 - T_0 + U) = \frac{\partial U}{\partial t}$$

an equation which applies to relative motion. When U does not contain t

$$T_2 - T_0 + U = h.$$

When U does contain t the equation

$$T_2 - T_0 = -U + \int \frac{\partial U}{\partial t}\, dt + h$$

is a purely formal integral because it is to be understood that any coordinates occurring in $\partial U/\partial t$ are expressed in terms of t before integration. This implies a knowledge of the complete solution of the problem. But the equation is not without its uses. Thus if $U = U_0 + U'$, where U_0 does not contain t and the effect of U' is small in comparison with the effect of U_0, preliminary values of the coordinates in terms of t may be found. When these are inserted in $\partial U'/\partial t$ a closer approximation to the true integral will be obtained and the process can be repeated. The true meaning of the equation is therefore connected with a method of approximation.

124. The above form (2) of the integral of energy is directly connected with the Hamiltonian form of the equations of motion whereby the n Lagrangian equations of the second order are replaced by a system of $2n$ equations of the first order. For we may write

$$\sum_r \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = H, \quad \frac{\partial L}{\partial \dot{q}_r} = p_r.$$

The n equations for p_r are linear in $\dot{q}_r$ and when solved express $\dot{q}_r$ in terms of (q_r, p_r), this symbol being used, where no ambiguity is to be feared, to denote all the quantities $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$. Hence L and H can be expressed either in terms of $(q_r, \dot{q}_r)$ or of (q_r, p_r). Thus

$$\delta L = \sum_r \frac{\partial L}{\partial q_r} . \delta q_r + \sum_r \frac{\partial L}{\partial \dot{q}_r} . \delta \dot{q}_r$$

$$\delta \Sigma \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} = \sum_r \dot{q}_r \, \delta p_r + \sum_r \frac{\partial L}{\partial \dot{q}_r} . \delta \dot{q}_r$$

and therefore

$$\delta H = \sum_r (\dot{q}_r \, \delta p_r - \dot{p}_r \, \delta q_r)$$

since

$$\dot{p}_r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \frac{\partial L}{\partial q_r}.$$

It follows that

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, \ldots, n) \quad \ldots\ldots\ldots\ldots\ldots(3)$$

and this is the form of the equations which is called *canonical.*

When L has its natural form, $H = T + U$. If L does not contain t explicitly, neither does H, and the integral of energy (2) becomes simply $H = h$.

125. Let us consider the differential form

$$d\theta = \sum_r p_r\, dq_r - H\, dt$$

or

$$d\left(\sum_r p_r q_r - \theta\right) = \sum_r q_r\, dp_r + H\, dt.$$

If $d\theta$ is a perfect differential, the right-hand side of both equations must also be perfect differentials, and this requires that

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad \frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}$$

or the canonical equations must be satisfied. Let us suppose now a transformation from the variables (q_r, p_r) to the variables (Q_r, P_r) such that

$$\sum_r P_r\, dQ_r - \sum_r p_r\, dq_r = -dW \quad \text{.........................(4)}$$

where dW is a perfect differential and W is expressible either in terms of (q_r, p_r) or of (Q_r, P_r). Such a transformation is called a *contact transformation*, or in the particular case when (q_r) can be expressed in terms of (Q_r) alone [by relations not involving (p_r) or (P_r)] an *extended point transformation*. If W contains t in addition we may write

$$\sum_r P_r\, dQ_r - \sum_r p_r\, dq_r - \frac{\partial W}{\partial t}\,.\, dt = -dW - \frac{\partial W}{\partial t}\,.\, dt$$

so that when $d\theta$ is introduced

$$\sum_r P_r\, dQ_r - \left(H + \frac{\partial W}{dt}\right) dt = d\theta - dW - \frac{\partial W}{\partial t}\,.\, dt.$$

Each side of this equation is a perfect differential provided $d\theta$ is a perfect differential, and in this case

$$\dot{P}_r = -\frac{\partial K}{\partial Q_r}, \quad \dot{Q}_r = \frac{\partial K}{\partial P_r} \quad \text{............................(5)}$$

where

$$K = H + \frac{\partial W}{\partial t} \quad \text{..................................(6)}$$

Since these equations equally with the form (3) express the conditions required if $d\theta$ is to be a perfect differential, they must be equivalent to (3). Thus we see that any transformation of variables satisfying the condition (4) leaves the equations of motion in the canonical form.

126. In consequence of (4)

$$P_r = -\frac{\partial W}{\partial Q_r}, \quad p_r = \frac{\partial W}{\partial q_r} \quad \text{............................(7)}$$

Hence K will vanish in virtue of (6) provided

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) + \frac{\partial W}{\partial t} = 0 \quad \text{..................(8)}$$

This equation is known as the Hamilton-Jacobi equation. But when $K=0$,

$$P_r = \beta_r, \quad Q_r = \alpha_r$$

where α_r and β_r, by (5), are arbitrary constants. Hence if any function W can be found which satisfies (8) and contains n arbitrary constants (α_r) in addition to (q_r) and t, the solution of the problem is completely expressed by the $2n$ equations (7) written in the form

$$\frac{\partial W}{\partial \alpha_r} = -\beta_r, \quad p_r = \frac{\partial W}{\partial q_r} \qquad \text{.........................(9)}$$

where (β_r) are n additional arbitrary constants.

If H does not contain t explicitly we may write

$$W = -\alpha_n t + W'$$

where W' is a solution, containing $(n-1)$ constants (α_r) apart from α_n but not t, of the equation

$$H\left(q_1, \ldots, q_n, \frac{\partial W'}{\partial q_1}, \ldots, \frac{\partial W'}{\partial q_n}\right) = \alpha_n \qquad \text{..................(10)}$$

The solution (9) is therefore replaced by

$$\left.\begin{aligned} &\frac{\partial W'}{\partial \alpha_r} = -\beta_r, \quad p_r = \frac{\partial W'}{\partial q_r}, \; (r = 1, 2, \ldots, n-1) \\ &\frac{\partial W'}{\partial \alpha_n} = t - \beta_n, \quad p_n = \frac{\partial W'}{\partial q_n} \end{aligned}\right\} \qquad \text{.........(11)}$$

127. In the set of equations (7) W is an arbitrary function of (Q_r, q_r). Instead of making W a solution of (8) let it satisfy the equation

$$H_0\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) + \frac{\partial W}{\partial t} = 0$$

where H_0 is the Hamiltonian function of another problem also presenting n degrees of freedom. Hence as before

$$P_r = \beta_r, \quad Q_r = \alpha_r$$

where (α_r, β_r) are the $2n$ arbitrary constants of the problem defined by H_0. Hence the equations (5) and (6) become

$$\dot{\alpha}_r = \frac{\partial K}{\partial \beta_r}, \quad \dot{\beta}_r = -\frac{\partial K}{\partial \alpha_r} \qquad \text{...........................(12)}$$

where

$$K = H + \frac{\partial W}{\partial t} = H - H_0.$$

Thus if the H_0 problem has been solved and the constants of a solution of the corresponding Hamilton-Jacobi equation are known, the same form of solution applies to the H problem with the difference that the quantities which remain constant in the first problem undergo variations in the second

problem which are defined by (12). This is the foundation of Lagrange's method of the *variation of arbitrary constants.* The simple form of (12) depends essentially on the function K being expressed in terms of the constants which occur in a solution of a Hamilton-Jacobi equation and which may be called a set of *canonical* constants.

If we suppose that the problem defined by H_0 has been solved by some other method than through the medium of a Hamilton-Jacobi equation, a different set of constants will be obtained. Let A_m be a typical member of such a set. Then A_m is some function of (α_r, β_r). Hence

$$\begin{aligned}\dot{A}_m &= \sum_r \frac{\partial A_m}{\partial \alpha_r} . \dot{\alpha}_r + \sum_r \frac{\partial A_m}{\partial \beta_r} . \dot{\beta}_r \\ &= \sum_r \left(\frac{\partial A_m}{\partial \alpha_r} . \frac{\partial K}{\partial \beta_r} - \frac{\partial A_m}{\partial \beta_r} . \frac{\partial K}{\partial \alpha_r}\right) \\ &= \sum_r \sum_s \left(\frac{\partial A_m}{\partial \alpha_r} . \frac{\partial K}{\partial A_s} . \frac{\partial A_s}{\partial \beta_r} - \frac{\partial A_m}{\partial \beta_r} . \frac{\partial K}{\partial A_s} . \frac{\partial A_s}{\partial \alpha_r}\right) \\ &= \sum_s \{A_m, A_s\} \frac{\partial K}{\partial A_s}\end{aligned}$$

where $K = H - H_0$ as before, and

$$\{A_m, A_s\} = \sum_r \left(\frac{\partial A_m}{\partial \alpha_r} . \frac{\partial A_s}{\partial \beta_r} - \frac{\partial A_m}{\partial \beta_r} . \frac{\partial A_s}{\partial \alpha_r}\right)$$

a form of expression which will be defined later (§ 130) as a Poisson's bracket.

128. Let us consider the integral

$$\begin{aligned}J &= \int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1} (T - U)\,dt \\ &= \int_{t_0}^{t_1} (-H + \Sigma p_r \dot{q}_r)\,dt \quad \text{.........................(13)}\end{aligned}$$

by the first set of equations in § 124. We have therefore

$$\begin{aligned}\delta J &= \int_{t_0}^{t_1} (-\delta H + \Sigma \dot{q}_r \delta p_r + \Sigma p_r \delta \dot{q}_r)\,dt \\ &= \Big[\Sigma p_r \delta q_r\Big]_0^1 + \int_{t_0}^{t_1} (-\delta H + \Sigma \dot{q}_r \delta p_r - \Sigma \dot{p}_r \delta q_r)\,dt\end{aligned}$$

where δ denotes a change in (q_r, p_r) but leaves t at each point unaltered. Hence $\delta J = 0$ if $\delta q_r = 0$ at the limits and if the canonical equations are satisfied. And this proves *Hamilton's principle* that in the passage from one fixed configuration to another the integral J has a stationary value for the actual motion as compared with any other neighbouring motion in which the time at corresponding points is the same.

If however δ denotes a change in t,

$$\delta J = -\delta \int_{t_0}^{t_1} H dt + \delta \int_0^1 \Sigma p_r dq_r$$

$$= -\Big[H \delta t \Big]_0^1 .$$

Hence when two neighbouring forms of motion, each compatible with the canonical equations, are compared, the complete variation between two positions 0 and 1 is

$$\delta J = \Big[\Sigma p_r \delta q_r \Big]_0^1 - \Big[H \, \delta t \Big]_0^1 .$$

Accordingly, if the initial time is taken as fixed and (α_r, β_r) are the initial values of (q_r, p_r), we have

$$\frac{\partial J}{\partial q_r} = p_r, \quad \frac{\partial J}{\partial \alpha_r} = -\beta_r$$

and

$$\frac{\partial J}{\partial t} = -H(q_r, p_r) = -H\left(q_r, \frac{\partial J}{\partial q_r}\right).$$

But this is the Hamilton-Jacobi equation. Hence the integral J is a particular solution of this equation. And further, since we have reproduced the equations (8) and (9) of § 126 except that J is written in the place of W, we see that J is that solution which contains the initial values of the coordinates as its n arbitrary constants.

129. Let us suppose now that H does not contain t explicitly, so that the integral of energy $H = h$ exists. Then if

$$J = \int_{t_0}^{t_1} \Sigma p_r \dot{q}_r dt = \int_{t_0}^{t_1} (L + h) \, dt \quad \text{.........................(14)}$$

$$\delta J = \Big[\Sigma p_r \delta q_r \Big]_0^1 + \int_{t_0}^{t_1} (\Sigma \dot{q}_r \delta p_r - \Sigma \dot{p}_r \delta q_r) \, dt.$$

But

$$\Sigma \dot{q}_r \delta p_r - \Sigma \dot{p}_r \delta q_r = \Sigma \frac{\partial H}{\partial p_r} \delta p_r + \Sigma \frac{\partial H}{\partial q_r} \delta q_r$$

$$= \delta h$$

and therefore

$$\delta J = \Big[\Sigma p_r \delta q_r \Big]_0^1 + \int_{t_0}^{t_1} \delta h \, . \, dt.$$

This is the complete variation of J and it vanishes between fixed terminal points if $\delta h = 0$ in each intermediate position, i.e. if the time is assigned to each displaced position in such a way that the equation $H = h$ is satisfied in the varied motion. Under these conditions the integral

$$J = \int_{t_0}^{t_1} (L + h) \, dt = \int_{t_0}^{t_1} (T - U + h) \, dt$$

has a stationary value in the course of the actual motion as compared with motion in any neighbouring paths.

This integral is called the *action* and the proposition established is known as the *principle of least action.* When T is a quadratic function of the velocities $h = T + U$ and the integral becomes

$$J = 2\int_{t_0}^{t_1} T dt \quad \text{.....................(15)}$$

and in problems which involve only one material particle this is simply

$$J = \int_{t_0}^{t_1} v^2 dt = \int_0^1 v ds \quad \text{.....................(16)}$$

where v is the velocity of the particle (of unit mass).

The integrals which we have found to be stationary are not necessarily minima. The necessary conditions in order that an integral

$$J = \int_{t_0}^{t_1} f(q_r, \dot{q}_r)\, dt$$

shall be an actual minimum are:

(1) The first variation δJ vanishes between fixed terminal points.

(2) The function of (ϵ_r)

$$\phi(\epsilon_r) = f(q_r, \dot{q}_r + \epsilon_r) - \Sigma \epsilon_r \frac{\partial f}{\partial \dot{q}_r}$$

is a minimum.

(3) Between the terminal positions 0 and 1 no intermediate position P exists such that 0 and P can be joined by a neighbouring path which satisfies the dynamical conditions and is other than the path considered. The nearest point to 0 on the path which does not satisfy this condition is called the *kinetic focus* of the point 0.

130. It is necessary to study the properties of certain expressions connected with the transformations which are frequently employed. Let $u_1, u_2, \ldots, u_{2n}$ be $2n$ distinct functions of (q_r, p_r). The first expression is

$$\sum_r \left(\frac{\partial q_r}{\partial u_l} \cdot \frac{\partial p_r}{\partial u_m} - \frac{\partial q_r}{\partial u_m} \cdot \frac{\partial p_r}{\partial u_l}\right) = \sum_r \frac{\partial (q_r, p_r)}{\partial (u_l, u_m)} \quad \text{...............(17)}$$

which is called a *Lagrange's bracket* and is denoted by $[u_l, u_m]$. The second expression is

$$\sum_r \left(\frac{\partial u_l}{\partial q_r} \cdot \frac{\partial u_m}{\partial p_r} - \frac{\partial u_m}{\partial q_r} \cdot \frac{\partial u_l}{\partial p_r}\right) = \sum_r \frac{\partial (u_l, u_m)}{\partial (q_r, p_r)} \quad \text{...............(18)}$$

This is called a *Poisson's bracket* and will be denoted here by the symbol $\{u_l, u_m\}$. It is evident that we have

$$[u_l, u_m] = -[u_m, u_l], \quad (l \neq m)$$

$$\{u_l, u_m\} = -\{u_m, u_l\}, \quad (l \neq m)$$

$$[u_l, u_l] = \{u_l, u_l\} = 0.$$

There are also relations between the two types of expression, and these we shall now investigate.

Let two linear substitutions be defined by

$$x_l = \sum_r^n \frac{\partial q_r}{\partial u_l} \cdot y_r + \sum_r^n \frac{\partial p_r}{\partial u_l} \cdot y_{n+r}$$

and

$$y_r = \sum_m^{2n} \frac{\partial p_r}{\partial u_m} \cdot z_m, \quad y_{n+r} = -\sum_m^{2n} \frac{\partial q_r}{\partial u_m} \cdot z_m$$

where r can have all values $1, \ldots, n$ and l and m can have all values $1, \ldots, 2n$. The result of eliminating y_r, y_{n+r} is to give

$$x_l = \sum_m^{2n} z_m \sum_r^n \left(\frac{\partial q_r}{\partial u_l} \cdot \frac{\partial p_r}{\partial u_m} - \frac{\partial p_r}{\partial u_l} \cdot \frac{\partial q_r}{\partial u_m}\right)$$

$$= \sum_m^{2n} [u_l, u_m] z_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

But the substitutions can be reversed by writing

$$y_r = \sum_l^{2n} \frac{\partial u_l}{\partial q_r} \cdot x_l, \quad y_{n+r} = \sum_l^{2n} \frac{\partial u_l}{\partial p_r} \cdot x_l$$

$$z_m = \sum_r^n \frac{\partial u_m}{\partial p_r} \cdot y_r - \sum_r^n \frac{\partial u_m}{\partial q_r} \cdot y_{n+r}.$$

The equivalence of these forms is easily verified since

$$\sum_l^{2n} \left[\frac{\partial u_l}{\partial q_r} \cdot \frac{\partial q_r}{\partial u_l}\right] = 1, \quad \sum_l^{2n} \left[\frac{\partial u_l}{\partial q_r} \cdot \frac{\partial p_r}{\partial u_l}\right] = 0, \ldots.$$

When y_r, y_{n+r} are eliminated, these give

$$z_m = \sum_l^{2n} x_l \sum_r^n \left(\frac{\partial u_l}{\partial q_r} \cdot \frac{\partial u_m}{\partial p_r} - \frac{\partial u_m}{\partial q_r} \cdot \frac{\partial u_l}{\partial p_r}\right)$$

$$= \sum_l^{2n} \{u_l, u_m\} x_l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

The resultant substitutions (19) and (20) must therefore be equivalent, and accordingly their determinants, written in the forms

$$\begin{vmatrix} [u_1, u_1], [u_1, u_2], \ldots, [u_1, u_{2n}] \\ [u_2, u_1], [u_2, u_2], \ldots, [u_2, u_{2n}] \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ [u_{2n}, u_1], [u_{2n}, u_2], \ldots, [u_{2n}, u_{2n}] \end{vmatrix} \text{ and } \begin{vmatrix} \{u_1, u_1\}, \{u_1, u_2\}, \ldots, \{u_1, u_{2n}\} \\ \{u_2, u_1\}, \{u_2, u_2\}, \ldots, \{u_2, u_{2n}\} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \{u_{2n}, u_1\}, \{u_{2n}, u_2\}, \ldots, \{u_{2n}, u_{2n}\} \end{vmatrix} \quad (21)$$

are reciprocal. This means that any constituent of either determinant is equal to the co-factor of the corresponding constituent in the other determinant divided by that determinant. Any Lagrange's bracket is thus expressible in terms of Poisson's brackets, and vice versa.

131. Let us now consider the explicit conditions for a contact transformation. We have in this case

$$\sum_r P_r\, dQ_r - \sum_r p_r dq_r = \sum_r P_r\, dQ_r - \sum_r \sum_l p_r \left(\frac{\partial q_r}{\partial Q_l} \cdot dQ_l + \frac{\partial q_r}{\partial P_l} \cdot dP_l\right)$$

a perfect differential. Hence

$$\frac{\partial}{\partial P_m}\left(\sum_r p_r \frac{\partial q_r}{\partial P_l}\right) = \frac{\partial}{\partial P_l}\left(\sum_r p_r \frac{\partial q_r}{\partial P_m}\right)$$

$$\frac{\partial}{\partial Q_m}\left(\sum_r p_r \frac{\partial q_r}{\partial Q_l}\right) = \frac{\partial}{\partial Q_l}\left(\sum_r p_r \frac{\partial q_r}{\partial Q_m}\right)$$

always, and

$$\frac{\partial}{\partial P_m}\left(\sum_r p_r \frac{\partial q_r}{\partial Q_l}\right) = \frac{\partial}{\partial Q_l}\left(\sum_r p_r \frac{\partial q_r}{\partial P_m}\right)$$

unless $l = m$, in which case

$$\frac{\partial}{\partial P_l}\left(\sum_r p_r \frac{\partial q_r}{\partial Q_l} - P_l\right) = \frac{\partial}{\partial Q_l}\left(\sum_r p_r \frac{\partial q_r}{\partial P_l}\right).$$

It is at once evident that these conditions may be written

$$[P_l,\, P_m] = 0, \quad [Q_l,\, Q_m] = 0$$

for all values of l and m,

$$[Q_l,\, P_m] = 0$$

for all unequal values of l and m, and

$$[Q_l,\, P_l] = 1$$

for all values of l. In other words, in the case of a contact transformation all the Lagrange's brackets vanish with the exception of those which are of the form $[Q_l, P_l]$, and these are all unity.

Let us now put

$$u_r = Q_r, \quad u_{n+r} = P_r, \quad (r = 1, 2, \ldots, n).$$

Then the substitution (19) becomes simply

$$x_r = z_{n+r}, \quad x_{n+r} = -z_r.$$

But this shows that all the Poisson's brackets occurring in (20) vanish except those which are of the form $\{u_l, u_{l\pm n}\}$, and these may be written

$$\{Q_l,\, P_l\} = 1 \text{ or } \{P_l,\, Q_l\} = -1.$$

The conditions for a contact transformation are therefore of the same simple form whether expressed in terms of Lagrange's or of Poisson's brackets.

Again, the substitutions of § 130,

$$x_l = \sum_r^n \frac{\partial q_r}{\partial u_l} y_r + \sum_r^n \frac{\partial p_r}{\partial u_l} y_{n+r}$$

$$z_m = \sum_r^n \frac{\partial u_m}{\partial p_r} y_r - \sum_r^n \frac{\partial u_m}{\partial q_r} y_{n+r}$$

become identical when $m = n + l$, since $z_{n+l} = x_l$. Hence

$$\frac{\partial q_r}{\partial Q_l} = \frac{\partial P_l}{\partial p_r}, \quad \frac{\partial p_r}{\partial Q_l} = -\frac{\partial P_l}{\partial q_r}.$$

But when $l = n + m$, they are identical except for an opposite sign throughout, since $x_{n+m} = -z_m$, and thus

$$\frac{\partial q_r}{\partial P_m} = -\frac{\partial Q_m}{\partial p_r}, \quad \frac{\partial p_r}{\partial P_m} = \frac{\partial Q_m}{\partial q_r}.$$

These relations hold for all values of l, m or r not exceeding n.

132. Let us consider the transformation

$$Q_r = q_r + \epsilon q_r', \quad P_r = p_r + \epsilon p_r'$$

where q_r', p_r' are any functions of (q_r, p_r) and ϵ is an infinitesimal constant.

If the transformation is an infinitesimal contact transformation,

$$\begin{aligned} dW &= \sum_r \{(p_r + \epsilon p_r')\, d(q_r + \epsilon q_r') - p_r dq_r\} \\ &= \epsilon \sum_r (p_r' dq_r + p_r dq_r') \end{aligned}$$

is a perfect differential. Hence we may write

$$\begin{aligned} \epsilon \sum_r (p_r' dq_r - q_r' dp_r) &= d\,(W - \epsilon \sum_r p_r q_r') \\ &= -\epsilon\,.\, dK \end{aligned}$$

where K may be any function of (q_r, p_r). Accordingly

$$q_r' = \frac{\partial K}{\partial p_r}, \quad p_r' = -\frac{\partial K}{\partial q_r}$$

and the general form of an infinitesimal contact transformation is given by

$$Q_r = q_r + \epsilon \frac{\partial K}{\partial p_r}, \quad P_r = p_r - \epsilon \frac{\partial K}{\partial q_r} \quad \text{..................(22)}$$

where K is an arbitrary function of (q_r, p_r).

If for ϵ we write δt, the equations (22) become

$$\frac{\delta q_r}{\delta t} = \frac{\partial K}{\partial p_r}, \quad \frac{\delta p_r}{\delta t} = -\frac{\partial K}{\partial q_r}$$

and comparing this form with that of the canonical equations of motion we see that the progressive motion of a system from point to point corresponds to a succession of infinitesimal contact transformations.

The effect of substituting (Q_r, P_r) in any function f of (q_r, p_r) is to produce an increment

$$\begin{aligned} \Delta f &= \sum_r \frac{\partial f}{\partial q_r}\,.\,\epsilon \frac{\partial K}{\partial p_r} - \sum_r \frac{\partial f}{\partial p_r}\,.\,\epsilon \frac{\partial K}{\partial q_r} \\ &= \epsilon\,\{f, K\} \quad \text{..(23)} \end{aligned}$$

133. Let us consider a disturbed motion in which (q_r, p_r) become $(q_r + \delta q_r, p_r + \delta p_r)$ at the time t. If this motion is compatible with the canonical equations

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}$$

we must have

$$\frac{d}{dt}(\delta q_r) = \sum_s \left(\frac{\partial^2 H}{\partial p_r \partial q_s} . \delta q_s + \frac{\partial^2 H}{\partial p_r \partial p_s} . \delta p_s\right)$$

with similar equations for δp_r. Now let us suppose that the new variables are those given by (22). These will lead to a particular solution of the varied motion provided

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial K}{\partial p_r}\right) &= \sum_s \left(\frac{\partial^2 H}{\partial p_r \partial q_s} . \frac{\partial K}{\partial p_s} - \frac{\partial^2 H}{\partial p_r \partial p_s} . \frac{\partial K}{\partial q_s}\right) \\
&= \frac{\partial}{\partial p_r} \sum_s \left(\frac{\partial H}{\partial q_s} . \frac{\partial K}{\partial p_s} - \frac{\partial H}{\partial p_s} . \frac{\partial K}{\partial q_s}\right) \\
&\quad - \sum_s \left(\frac{\partial H}{\partial q_s} . \frac{\partial^2 K}{\partial p_r \partial p_s} - \frac{\partial H}{\partial p_s} . \frac{\partial^2 K}{\partial p_r \partial q_s}\right) \\
&= \frac{\partial}{\partial p_r} \sum_s \left(-\dot{p}_s \frac{\partial K}{\partial p_s} - \dot{q}_s \frac{\partial K}{\partial q_s}\right) \\
&\quad + \sum_s \left(\dot{p}_s \frac{\partial^2 K}{\partial p_r \partial p_s} + \dot{q}_s \frac{\partial^2 K}{\partial p_r \partial q_s}\right) \\
&= \frac{\partial}{\partial p_r}\left(\frac{\partial K}{\partial t} - \frac{dK}{dt}\right) + \frac{d}{dt}\left(\frac{\partial K}{\partial p_r}\right) - \frac{\partial}{\partial t}\left(\frac{\partial K}{\partial p_r}\right)
\end{aligned}$$

or

$$0 = -\frac{\partial}{\partial p_r}\left(\frac{dK}{dt}\right)$$

with a similar set of conditions arising from the equations for δp_r. But it is evident that all these conditions will be satisfied if K is an integral of the system, for then $\dot{K} = 0$. We thus see that if K is an integral, the equations (22) are a particular solution of the equations for the disturbed motion.

134. Let u be another integral of the undisturbed system. Then $u + \Delta u$ must also have a constant value in the disturbed motion. But by (23)

$$\Delta u = \epsilon \{u, K\}$$

when the disturbed motion is that obtained by the infinitesimal contact transformation derived from K. Hence $\{u, K\}$ must be constant, and we have Poisson's theorem: if u and K are two integrals of a system, the Poisson's bracket $\{u, K\}$ is also an integral. It might be supposed that a knowledge of two integrals would thus lead to the discovery of all the

integrals of a problem. This is not so in general. The known integrals are more often of a generic type, particularly in the case of those gravitational problems with which we have to deal, and fall into closed groups. For example, if we start from two integrals of area we obtain by Poisson's theorem the third integral of the same type and no further progress can be made in this way. In order to obtain fresh information it is necessary to start from integrals which are special to the problem considered.

Let $u_1, u_2, \ldots, u_{2n}$ be $2n$ distinct integrals of the problem. Then each Poisson's bracket of the type $\{u_r, u_s\}$ is constant throughout the motion. But we have seen in § 130 that a Lagrange's bracket $[u_r, u_s]$ can be expressed in terms of all the Poisson's brackets. Hence $[u_r, u_s]$ is also constant throughout the motion. But this gives no means of finding additional integrals of the problem, for in order to calculate $[u_r, u_s]$ it is first necessary to express (q_r, p_r) in terms of the $2n$ integrals (u_r). And this presupposes that the problem has been completely solved.

CHAPTER XIII

VARIATION OF ELEMENTS

135. The Hamilton-Jacobi equation corresponding to elliptic motion about a fixed centre of attraction is very simply solved when the variables are expressed in polar coordinates (r, l, λ), so that $(l, \lambda$ having the same relation to one another as longitude and latitude)

$$q_1 = r, \quad q_2 = \lambda, \quad q_3 = l.$$

Then, after suppressing the factor m in the potential energy U and therefore treating the mass factor in the momenta as unity,

$$\begin{aligned} U &= -\mu r^{-1}, \quad \mu = k^2(1+m) = n^2a^3 \\ 2T &= \dot{r}^2 + r^2\dot{\lambda}^2 + r^2\cos^2\lambda \,.\, \dot{l}^2 \\ p_1 &= \dot{r}, \quad p_2 = r^2\dot{\lambda}, \quad p_3 = r^2\cos^2\lambda \,.\, \dot{l} \\ H &= T + U = \tfrac{1}{2}(p_1^2 + r^{-2}p_2^2 + r^{-2}\sec^2\lambda \,.\, p_3^2) - \mu r^{-1}. \end{aligned}$$

The Hamilton-Jacobi equation (§ 126) therefore takes the form, since H does not contain t,

$$\left(\frac{\partial W'}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W'}{\partial \lambda}\right)^2 + \frac{1}{r^2\cos^2\lambda}\left(\frac{\partial W'}{\partial l}\right)^2 = 2\alpha_1 + \frac{2\mu}{r}$$

where $W = W' - \alpha_1 t$. Integration by separation of the variables is then easy. For

$$\left(\frac{\partial W'}{\partial l}\right)^2 = \alpha_3^2, \quad \left(\frac{\partial W'}{\partial \lambda}\right)^2 = \alpha_2^2 - \alpha_3^2\sec^2\lambda$$

$$\left(\frac{\partial W'}{\partial r}\right)^2 = 2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}$$

obviously satisfy the equation. Hence

$$W' = \int_{r_0}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}} dr + \int_0^{\lambda}(\alpha_2^2 - \alpha_3^2\sec^2\lambda)^{\frac{1}{2}}\, d\lambda + \alpha_3 l$$

is an integral which contains the three independent constants α_1, α_2, α_3. Therefore the complete solution of the problem is given by the equations

$$t-\beta_1=\frac{\partial W'}{\partial \alpha_1}=\int_{r_0}^{r}\left(2\alpha_1+\frac{2\mu}{r}-\frac{\alpha_2^2}{r^2}\right)^{-\frac{1}{2}}dr$$

$$-\beta_2=\frac{\partial W'}{\partial \alpha_2}=-\int_{r_0}^{r}\frac{\alpha_2}{r^2}\left(2\alpha_1+\frac{2\mu}{r}-\frac{\alpha_2^2}{r^2}\right)^{-\frac{1}{2}}dr+\int_0^{\lambda}\alpha_2\left(\alpha_2^2-\alpha_3^2\sec^2\lambda\right)^{-\frac{1}{2}}d\lambda$$

$$-\beta_3=\frac{\partial W'}{\partial \alpha_3}=l-\int_0^{\lambda}\alpha_3\sec^2\lambda\left(\alpha_2^2-\alpha_3^2\sec^2\lambda\right)^{-\frac{1}{2}}d\lambda$$

where β_1, β_2, β_3 are three additional constants. The lower limit r_0 is also arbitrary. It may be identified with the pericentric distance, and then the integrals depending on r will vanish at the pericentre.

136. We have now to determine the meaning of the six constants of integration. Since the integral in the first equation vanishes at perihelion, β_1 is clearly the time at this point. Also, by the same equation,

$$\dot{r}^2=\frac{2\mu}{r}-\frac{\alpha_2^2}{r^2}+2\alpha_1$$

$$=2\alpha_1(r-r_1)(r-r_2)/r^2.$$

But at an apse, $\dot{r}=0$ and $r=a(1\pm e)$. These then are the values of r_1, r_2, and hence

$$\mu=-2a\alpha_1,\quad \alpha_2^2=-2a^2(1-e^2)\alpha_1$$

or

$$\alpha_1=-\mu/2a,\quad \alpha_2=\sqrt{\{\mu a(1-e^2)\}}.$$

Also if we put $\alpha_3/\alpha_2=\cos i$ the second and third equations become on integration

$$-\beta_2=-f_1(r)+\sin^{-1}(\sin\lambda/\sin i)$$

$$-\beta_3=\quad l-\sin^{-1}(\tan\lambda/\tan i)$$

or

$$\sin\lambda=\sin i\sin\{f_1(r)-\beta_2\}$$

$$\tan\lambda=\tan i\sin(l+\beta_3).$$

This last equation shows that the motion takes place in a fixed plane making the angle i with the plane $\lambda=0$, which may be taken to represent, for example, the ecliptic, with l and λ as the longitude and latitude of the planet. Thus the meaning of $\alpha_3=\alpha_2\cos i$ is defined, and $-\beta_3$ is simply the longitude of the node. The preceding equation then shows that $f_1(r)-\beta_2$ is the angle between the radius vector of the planet and the line of nodes, i.e. the argument of latitude. But at perihelion the integral $f_1(r)$ vanishes. Hence $-\beta_2$ is simply the angle in the orbit from the node to perihelion, or $\varpi-\Omega$ in the ordinary notation. The canonical elements which we

have introduced can therefore be expressed in terms of the usual elements (T being reckoned from the epoch when the mean longitude is ϵ) thus:

$$\begin{aligned} \alpha_1 &= -\mu/2a, & \beta_1 &= T = -(\epsilon - \varpi)/n \\ \alpha_2 &= \sqrt{\{\mu a(1-e^2)\}}, & \beta_2 &= -\varpi + \Omega \\ \alpha_3 &= \sqrt{\{\mu a(1-e^2)\}}\cos i, & \beta_3 &= -\Omega. \end{aligned}$$

The homogeneity of these constants will be increased by introducing $\alpha = \sqrt{\mu a}$ instead of α_1. This makes $2\alpha_1 = -\mu^2/\alpha^2$ and $W = W' + \mu^2 t/2\alpha^2$. Hence β_1 will be replaced by β, where

$$-\beta = \frac{\partial W}{\partial \alpha} = \frac{\partial W'}{\partial \alpha} - \frac{\mu^2 t}{\alpha^3} = \frac{\mu^2}{\alpha^3}\left(\frac{\partial W'}{\partial \alpha_1} - t\right)$$

$$= \frac{\mu^2}{\alpha^3}\left\{\int_{r_0}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{-\frac{1}{2}} dr - t\right\}.$$

Since the integral vanishes at perihelion, and $t = T$ at this point,

$$\beta = \frac{\mu^2 T}{\alpha^3} = \sqrt{\frac{\mu}{a^3}}\,.\,T = nT = -\epsilon + \varpi.$$

The other constants are easily seen not to be affected by the change in α_1, β_1, which can accordingly be replaced by

$$\alpha = \sqrt{(\mu a)}, \qquad \beta = nT = -\epsilon + \varpi$$

where ϵ is the mean longitude of the planet at the time $t = 0$.

137. The expressions for α, α_2, α_3, β, β_2, β_3 in terms of the ordinary elliptic elements which have just been found make it very easy to calculate the Lagrange's brackets

$$[u, v] = \Sigma\left(\frac{\partial \alpha}{\partial u}\cdot\frac{\partial \beta}{\partial v} - \frac{\partial \beta}{\partial u}\cdot\frac{\partial \alpha}{\partial v}\right)$$

where u, v are any pair of the six elements a, e, i, Ω, ϖ, ϵ. Since α, α_2, α_3 are functions of a, e, i alone and β, β_2, β_3 are functions of Ω, ϖ, ϵ alone, the Lagrange's bracket for any pair of either set of three elements vanishes. It is equally evident on inspection that $[e, \epsilon]$, $[i, \varpi]$ and $[i, \epsilon]$ also vanish, the two constituents never occurring in a corresponding pair of canonical constants. Hence the complete array of Lagrange's brackets may be set out thus:

	a	e	i	Ω	ϖ	ϵ
a	0	0	0	$[a, \Omega]$	$[a, \varpi]$	$[a, \epsilon]$
e	0	0	0	$[e, \Omega]$	$[e, \varpi]$	0
i	0	0	0	$[i, \Omega]$	0	0
Ω	$-[a, \Omega]$	$-[e, \Omega]$	$-[i, \Omega]$	0	0	0
ϖ	$-[a, \varpi]$	$-[e, \varpi]$	0	0	0	0
ϵ	$-[a, \epsilon]$	0	0	0	0	0

where the first constituent of each bracket taken positively is placed in the column on the left and the second constituent in the line at the top. The brackets in the second diagonal really contain only one term and are at once seen to be

$$[a, \epsilon] = -\tfrac{1}{2}\sqrt{\mu/a}$$
$$[e, \varpi] = e\sqrt{\mu a}/\sqrt{(1-e^2)}$$
$$[i, \Omega] = \sqrt{\mu a(1-e^2)}\,.\sin i$$

while the remaining three brackets contain two terms and are

$$[a, \Omega] = \tfrac{1}{2}\sqrt{(1-e^2)\mu/a}\,(1-\cos i)$$
$$[a, \varpi] = \tfrac{1}{2}\sqrt{\mu/a}\,.(1-\sqrt{1-e^2})$$
$$[e, \Omega] = -e\sqrt{\mu a}\,(1-\cos i)/\sqrt{1-e^2}.$$

The value of the whole determinant depends simply on the constituents in the second diagonal and is evidently

$$\Delta = [a, \epsilon]^2 [e, \varpi]^2 [i, \Omega]^2$$
$$= \tfrac{1}{4}\mu^3 a e^2 \sin^2 i.$$

138. It is now easy to form the reciprocal determinant, the constituents of which are the Poisson's brackets of pairs of elements. On account of the large number of zeros in the above determinant a corresponding number of minors vanish and the rest can be calculated without difficulty. It can in fact be verified by simple inspection that the reciprocal determinant takes the form:

$$\begin{array}{c|cccccc|}
 & a & e & i & \Omega & \varpi & \epsilon \\
a & 0 & 0 & 0 & 0 & 0 & \{a, \epsilon\} \\
e & 0 & 0 & 0 & 0 & \{e, \varpi\} & \{e, \epsilon\} \\
i & 0 & 0 & 0 & \{i, \Omega\} & \{i, \varpi\} & \{i, \epsilon\} \\
\Omega & 0 & 0 & -\{i, \Omega\} & 0 & 0 & 0 \\
\varpi & 0 & -\{e, \varpi\} & -\{i, \varpi\} & 0 & 0 & 0 \\
\epsilon & -\{a, \epsilon\} & -\{e, \epsilon\} & -\{i, \epsilon\} & 0 & 0 & 0
\end{array}$$

the first constituent of each bracket (written positively) being indicated in the column on the left and the second constituent in the top line as before. It is also clear that the partial substitutions (§ 130)

$$x_1 = [a, \Omega]\,z_4 + [a, \varpi]\,z_5 + [a, \epsilon]\,z_6$$
$$x_2 = [e, \Omega]\,z_4 + [e, \varpi]\,z_5$$
$$x_3 = [i, \Omega]\,z_4$$

and

$$\begin{aligned}
z_4 &= \{i, \Omega\}\, x_3 \\
z_5 &= \{e, \varpi\}\, x_2 + \{i, \varpi\}\, x_3 \\
z_6 &= \{a, \epsilon\}\, x_1 + \{e, \epsilon\}\, x_2 + \{i, \epsilon\}\, x_3
\end{aligned}$$

must be equivalent, and it readily follows that

$$\begin{aligned}
\{a, \epsilon\} &= 1/[a, \epsilon] = -2\sqrt{a/\mu} \\
\{e, \varpi\} &= 1/[e, \varpi] = \sqrt{1-e^2}/e\sqrt{\mu a} \\
\{i, \Omega\} &= 1/[i, \Omega] = 1/\sqrt{\mu a\,(1-e^2)}\sin i \\
\{e, \epsilon\} &= -[a, \varpi]/[a, \epsilon]\,[e, \varpi] \\
&= (1-\sqrt{1-e^2})\sqrt{1-e^2}/e\sqrt{\mu a} \\
\{i, \varpi\} &= -[e, \Omega]/[e, \varpi]\,[i, \Omega] \\
&= (1-\cos i)/\sqrt{\mu a\,(1-e^2)}\sin i \\
\{i, \epsilon\} &= -\{[a, \Omega]\,[e, \varpi] - [e, \Omega]\,[a, \varpi]\}/[a, \epsilon]\,[e, \varpi]\,[i, \Omega] \\
&= (1-\cos i)/\sqrt{\mu a\,(1-e^2)}\sin i.
\end{aligned}$$

The six Poisson's brackets are thus all known.

139. A solution of the Hamilton-Jacobi equation, involving the six arbitrary constants α, α_2, α_3, β, β_2, β_3, has been found for the case of undisturbed elliptic motion relative to the Sun. When the action of the other planets is taken into account, the potential energy U becomes $U-R$, where R is the disturbing function and is expressed by (§ 23)

$$R = k^2 \Sigma\, m_i \left(\frac{1}{\Delta_i} - \frac{xx_i + yy_i + zz_i}{r_i^3}\right).$$

Hence H becomes $H_0 - R$ and consequently by § 127 the constants of the approximate problem are in the more complete problem subject to variations which are defined by the equations

$$\frac{d\alpha_r}{dt} = -\frac{\partial R}{\partial \beta_r}, \quad \frac{d\beta_r}{dt} = +\frac{\partial R}{\partial \alpha_r}.$$

Here R is supposed to be expressed in terms of the constants mentioned in § 136, which refer to the motion of the planet considered undisturbed, and the time as it occurs in the expression of the coordinates of the disturbing planets. When instead of the canonical constants arising in the solution of the Hamilton-Jacobi equation the ordinary elements of elliptic motion are employed, the equations for the variations are no longer of the above simple type, but take the more complicated form

$$\frac{dA_r}{dt} = -\sum_s \{A_r, A_s\} \frac{\partial R}{\partial A_s}$$

where A_r represents any one of such elements. Since we have found the expressions for all the Poisson's brackets, the equations for the variation of

the usual elliptic elements can at once be written down in an explicit form. They are as follows:

$$\frac{da}{dt}=2\sqrt{a/\mu}\,.\,\frac{\partial R}{\partial\epsilon}$$

$$\frac{de}{dt}=-\frac{\cot\phi}{\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial\varpi}-\frac{\tan\frac{1}{2}\phi\cos\phi}{\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial\epsilon}$$

$$\frac{di}{dt}=-\frac{1}{\cos\phi\sin i\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial\Omega}-\frac{\tan\frac{1}{2}i}{\cos\phi\sqrt{\mu a}}\,.\left(\frac{\partial R}{\partial\varpi}+\frac{\partial R}{\partial\epsilon}\right)$$

$$\frac{d\Omega}{dt}=\frac{1}{\cos\phi\sin i\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial i}$$

$$\frac{d\varpi}{dt}=\frac{\cot\phi}{\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial e}+\frac{\tan\frac{1}{2}i}{\cos\phi\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial i}$$

$$\frac{d\epsilon}{dt}=-2\sqrt{a/\mu}\,.\,\frac{\partial R}{\partial a}+\frac{\tan\frac{1}{2}\phi\cos\phi}{\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial e}+\frac{\tan\frac{1}{2}i}{\cos\phi\,.\sqrt{\mu a}}\,.\,\frac{\partial R}{\partial i}.$$

A slight simplification has been made by writing $\sin\phi$ in place of e in the coefficients of the partial differentials of R.

140. The above set of equations for the variations of the elements is fundamental. An important point must be noticed in regard to them. The variation of a entails a corresponding variation of n which is determined by the relation $n^2a^3=\mu$. Now the disturbing function R is a periodic function of the mean anomaly and is expressed in terms of circular functions of multiples of nt. Hence the derivative of R with respect to a would contain the same circular functions multiplied by t and this introduction of terms not purely periodic would be inconvenient. The difficulty is avoided by an artifice which should be carefully noted.

We consider n (as distinct from a) to occur only in the arguments of these periodic terms. Otherwise a is used explicitly or if it is more convenient to use n outside the arguments, n is simply a function of a given by $n^2a^3=\mu$. Now ϵ enters into R only in the form $nt+\epsilon$ through the mean anomaly, so that

$$\frac{\partial R}{\partial\epsilon}=\frac{1}{t}\left(\frac{\partial R}{\partial n}\right)_{a=\text{const.}}.$$

Hence

$$\begin{aligned}\frac{d\epsilon}{dt}&=-2\sqrt{a/\mu}\,.\,\frac{\partial R}{\partial a}+\dots\\&=-2\sqrt{a/\mu}\left\{\left(\frac{\partial R}{\partial a}\right)_{n=\text{const.}}+\frac{dn}{da}\left(\frac{\partial R}{\partial n}\right)_{a=\text{const.}}\right\}+\dots\\&=-2\sqrt{a/\mu}\left\{\left(\frac{\partial R}{\partial a}\right)_{n=\text{const.}}+t\frac{dn}{da}\frac{\partial R}{\partial\epsilon}\right\}+\dots\\&=-2\sqrt{a/\mu}\left(\frac{\partial R}{\partial a}\right)_{n=\text{const.}}-t\frac{dn}{da}\,.\,\frac{da}{dt}+\dots\end{aligned}$$

or

$$\frac{d\epsilon}{dt} + t\frac{dn}{dt} = -2\sqrt{a/\mu}\left(\frac{\partial R}{\partial a}\right)_{n=\text{const.}} + \ldots.$$

If then we take ϵ' instead of ϵ, where

$$\frac{d\epsilon}{dt} + t\frac{dn}{dt} = \frac{d\epsilon'}{dt}$$

or

$$\epsilon + nt = \epsilon' + \int n\,dt$$

the *form* of the above equations for the variations of the six elements will be unaltered, since

$$\frac{\partial R}{\partial \epsilon} = \frac{\partial R}{\partial \epsilon'}$$

but their natural meaning will be so far altered that (1) n in the mean anomaly is *not* to be varied in forming the derivative with respect to a, and (2) nt in the mean anomaly is to be replaced by $\int n\,dt$. The secular terms which would arise from the cause mentioned are thus avoided.

The value of n is deduced directly from the value of a, and we have

$$\int n\,dt = \mu^{\frac{1}{2}}\int a^{-\frac{3}{2}}\,dt.$$

If this integral be denoted by ρ we have also

$$\frac{d^2\rho}{dt^2} = -\tfrac{3}{2}\sqrt{\mu/a^5}\cdot\frac{da}{dt} = -\frac{3}{a^2}\cdot\frac{\partial R}{\partial \epsilon}$$

or

$$\rho = -3\iint \frac{1}{a^2}\frac{\partial R}{\partial \epsilon}\,dt^2$$

which gives the finite variation of this part of the mean longitude in the disturbed orbit.

141. When e (and therefore ϕ) is small, and this is commonly the case, the coefficients in the variations of e and ϖ which contain $\cot\phi$ as a factor become large. This gives rise to a difficulty which can be avoided by introducing the transformation

$$h_1 = e\sin\varpi, \quad k_1 = e\cos\varpi.$$

The result of making this change, which can be verified without difficulty, is to substitute for the corresponding pair of equations

$$\frac{dh_1}{dt} = \frac{\cos\phi}{\sqrt{\mu a}}\cdot\frac{\partial R}{\partial k_1} + \frac{k_1\tan\frac{1}{2}i}{\cos\phi\sqrt{\mu a}}\cdot\frac{\partial R}{\partial i} - \frac{h_1\cos\phi}{2\cos^2\frac{1}{2}\phi\sqrt{\mu a}}\cdot\frac{\partial R}{\partial \epsilon}$$

$$\frac{dk_1}{dt} = -\frac{\cos\phi}{\sqrt{\mu a}}\cdot\frac{\partial R}{\partial h_1} - \frac{h_1\tan\frac{1}{2}i}{\cos\phi\sqrt{\mu a}}\cdot\frac{\partial R}{\partial i} - \frac{k_1\cos\phi}{2\cos^2\frac{1}{2}\phi\sqrt{\mu a}}\cdot\frac{\partial R}{\partial \epsilon}.$$

Similarly, when the angle between the plane of the orbit and the plane of reference is small, a pair of coefficients in the variations of i and Ω become large, and the transformation

$$h_2 = \sin i \sin \Omega, \quad k_2 = \sin i \cos \Omega$$

is useful. The result, which can be verified with equal ease, is to replace the equations named by the pair

$$\frac{dh_2}{dt} = \frac{\cos i}{\cos\phi\sqrt{\mu a}} \cdot \frac{\partial R}{\partial k_2} - \frac{h_2 \cos i}{2\cos^2\frac{1}{2}i\cos\phi\sqrt{\mu a}} \cdot \left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon}\right)$$

$$\frac{dk_2}{dt} = -\frac{\cos i}{\cos\phi\sqrt{\mu a}} \cdot \frac{\partial R}{\partial h_2} - \frac{k_2 \cos i}{2\cos^2\frac{1}{2}i\cos\phi\sqrt{\mu a}} \cdot \left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon}\right).$$

142. Another form of the equations for the variations of the elements, in which the disturbing forces appear explicitly, is of great importance. Let S, T be the components of these forces in the plane of the orbit along the radius vector and perpendicular to it, and W the component normal to the plane. Let u be the argument of latitude and (λ, μ, ν) the direction cosines of the radius vector, so that (§ 65)

$$\lambda = \cos u \cos \Omega - \sin u \sin \Omega \cos i$$

$$\mu = \cos u \sin \Omega + \sin u \cos \Omega \cos i$$

$$\nu = \sin u \sin i.$$

The direction cosines of the transversal and of the normal to the plane may be written

$$\frac{\partial\lambda}{\partial u}, \ \frac{\partial\mu}{\partial u}, \ \frac{\partial\nu}{\partial u} \ \text{and} \ \frac{1}{\sin u}\frac{\partial\lambda}{\partial i}, \ \frac{1}{\sin u}\frac{\partial\mu}{\partial i}, \ \frac{1}{\sin u}\frac{\partial\nu}{\partial i}$$

which must satisfy the conditions

$$\Sigma\lambda^2 = \Sigma\left(\frac{\partial\lambda}{\partial u}\right)^2 = \frac{1}{\sin^2 u}\Sigma\left(\frac{\partial\lambda}{\partial i}\right)^2 = 1$$

$$\Sigma\left(\lambda\frac{\partial\lambda}{\partial u}\right) = \Sigma\left(\lambda\frac{\partial\lambda}{\partial i}\right) = \Sigma\left(\frac{\partial\lambda}{\partial u}\cdot\frac{\partial\lambda}{\partial i}\right) = 0.$$

If σ be any one of the elliptic elements, we have also

$$\frac{\partial R}{\partial\sigma} = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial\sigma} + \frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial\sigma} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial\sigma}.$$

But the component of the disturbing forces along the axis of x is

$$\frac{\partial R}{\partial x} = \lambda S + \frac{\partial\lambda}{\partial u}T + \frac{1}{\sin u}\frac{\partial\lambda}{\partial i}W.$$

Hence

$$\frac{\partial R}{\partial\sigma} = \Sigma\left(\lambda S + \frac{\partial\lambda}{\partial u}T + \frac{1}{\sin u}\frac{\partial\lambda}{\partial i}W\right)\frac{\partial(\lambda r)}{\partial\sigma}$$

$$= S\frac{\partial r}{\partial\sigma} + rT\Sigma\left(\frac{\partial\lambda}{\partial u}\cdot\frac{\partial\lambda}{\partial\sigma}\right) + \frac{rW}{\sin u}\Sigma\left(\frac{\partial\lambda}{\partial i}\cdot\frac{\partial\lambda}{\partial\sigma}\right)$$

by the conditions mentioned. Now

$$r = a(1 - e\cos E), \qquad \tan\tfrac{1}{2}w = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}E$$

$$u = \varpi - \Omega + w, \qquad E - e\sin E = nt + \epsilon - \varpi.$$

In accordance with § 140 we treat n, as it occurs implicitly in u, as independent of a, and replace nt by $\int n\,dt$.

Hence

$$\frac{\partial R}{\partial a} = S\frac{\partial r}{\partial a} = \frac{rS}{a}$$

$$\frac{\partial R}{\partial i} = \frac{rW}{\sin u}\Sigma\left(\frac{\partial\lambda}{\partial i}\right)^2 = rW\sin u$$

$$\frac{\partial R}{\partial\Omega} = rT\Sigma\frac{\partial\lambda}{\partial u}\left(\frac{\partial\lambda}{\partial\Omega} - \frac{\partial\lambda}{\partial u}\right) + \frac{rW}{\sin u}\Sigma\frac{\partial\lambda}{\partial i}\left(\frac{\partial\lambda}{\partial\Omega} - \frac{\partial\lambda}{\partial u}\right)$$

(since λ contains Ω both explicitly and implicitly through u)

$$= rT\left\{\Sigma\left(\frac{\partial\lambda}{\partial u}\frac{\partial\lambda}{\partial\Omega}\right) - 1\right\} + \frac{rW}{\sin u}\Sigma\left(\frac{\partial\lambda}{\partial i}\frac{\partial\lambda}{\partial\Omega}\right)$$

$$= rT(\cos i - 1) + \frac{rW}{\sin u}(-\sin u\cos u\sin i)$$

$$= -2rT\sin^2\tfrac{1}{2}i - rW\cos u\sin i.$$

The remaining elements enter into (λ, μ, ν) only implicitly through u, so that in their case

$$\frac{\partial R}{\partial\sigma} = S\frac{\partial r}{\partial\sigma} + rT\Sigma\left(\frac{\partial\lambda}{\partial u}\right)^2\cdot\frac{\partial u}{\partial\sigma} + \frac{rW}{\sin u}\Sigma\left(\frac{\partial\lambda}{\partial i}\cdot\frac{\partial\lambda}{\partial u}\right)\frac{\partial u}{\partial\sigma}$$

$$= S\frac{\partial r}{\partial\sigma} + rT\left(\frac{\partial\varpi}{\partial\sigma} + \frac{\partial w}{\partial\sigma}\right).$$

Hence

$$\frac{\partial R}{\partial\epsilon} = S\,.\,ae\sin E\frac{\partial E}{\partial\epsilon} + rT\frac{\partial w}{\partial E}\cdot\frac{\partial E}{\partial\epsilon}$$

$$= S\,.\,a^2e\sin E/r + aT\sin w/\sin E$$

$$= aS\tan\phi\sin w + aT\sec\phi\,(1 + e\cos w).$$

Since r and w are both functions of $\epsilon - \varpi$,

$$\frac{\partial R}{\partial\varpi} = rT - \frac{\partial R}{\partial\epsilon}$$

and finally

$$\frac{\partial R}{\partial e} = S\frac{\partial r}{\partial e} + rT\frac{\partial w}{\partial e}$$

$$= aS\left(-\cos E + e\sin E\frac{\partial E}{\partial e}\right) + rT\left(\frac{\sin w}{\sin E}\frac{\partial E}{\partial e} + \frac{\sin w}{1-e^2}\right)$$

$$= aS\left(-\cos E + \frac{e\sin^2 E}{1-e\cos E}\right) + rT\sin w\left(\frac{1}{1-e\cos E} + \frac{1}{1-e^2}\right)$$

$$= aS\,.\,\frac{e-\cos E}{1-e\cos E} + rT\sin w\left(\frac{1+e\cos w}{1-e^2} + \frac{1}{1-e^2}\right)$$

$$= -aS\cos w + rT\sin w\,(2+e\cos w)\sec^2\phi.$$

It only remains to carry the expressions found for the derivatives of R into the equations of § 139 for the variations of the elements. The results are as follows:

$$\frac{da}{dt} = 2\sqrt{a^3/\mu}\,\{S\tan\phi\sin w + T\sec\phi\,(1+e\cos w)\}$$

$$\frac{de}{dt} = \sqrt{a/\mu}\cos\phi\,\{S\sin w + T(\cos w + \cos E)\}$$

$$\frac{di}{dt} = rW\cos u/\cos\phi\sqrt{\mu a}$$

$$\frac{d\Omega}{dt} = rW\sin u/\cos\phi\sin i\sqrt{\mu a}$$

$$\frac{d\varpi}{dt} = \{-aS\cos^2\phi\cos w + rT\sin w(2+e\cos w) + rW\sin\phi\tan\tfrac{1}{2}i\sin u\}/\sin\phi\cos\phi\sqrt{\mu a}$$

$$\frac{d\epsilon}{dt} = -2rS/\sqrt{\mu a} + 2\sin^2\tfrac{1}{2}\phi\frac{d\varpi}{dt} + 2\cos\phi\sin^2\tfrac{1}{2}i\frac{d\Omega}{dt}.$$

From the first two equations we get for the variation of the parameter $p = a(1-e^2)$

$$\frac{dp}{dt} = \cos^2\phi\frac{da}{dt} - 2a\sin\phi\frac{de}{dt} = 2rT\cos\phi\sqrt{a/\mu}.$$

It has been convenient to derive the above important set of equations from those which involve the derivatives of the disturbing function. But their form would be the same if the components of the forces were not such as can be expressed as the differentials of a single function. Thus they hold, for example, in the case of elliptic motion disturbed by a resisting medium.

Since $n^2a^3 = \mu$ is constant, the equation for the variation of a may be replaced by

$$\frac{dn}{dt} = -3\{S\sin\phi\sin w + T(1+e\cos w)\}/a\cos\phi.$$

Also

$$\frac{d}{dt}(\epsilon - \varpi) = -2rS/\sqrt{(\mu a)} - \cos\phi\frac{d\varpi}{dt} + rW\sin u\tan\tfrac{1}{2}i/\sqrt{(\mu a)}$$

$$= \{(a\cos^2\phi\cos w - 2r\sin\phi)S - rT\sin w(2 + e\cos w)\}/\sin\phi\sqrt{(\mu a)}$$

which gives the variation of the mean anomaly,

$$\frac{dM}{dt} = \frac{d}{dt}(\epsilon - \varpi) + \int\frac{dn}{dt}dt$$

part of the variation of nt being included in ϵ as explained in § 140 and mentioned above.

143. It has been seen in § 139 how the canonical solution of the problem of undisturbed elliptic motion leads to the canonical equations appropriate to the form of motion which follows from the introduction of disturbing forces. With a slight change of notation,

$$\begin{aligned} L &= \alpha = \sqrt{(\mu a)}, & l &= nt - \beta = \epsilon - \varpi + nt \\ G &= \alpha_2 = \sqrt{\{\mu a(1-e^2)\}}, & g &= -\beta_2 = \varpi - \Omega \\ H &= \alpha_3 = \sqrt{\{\mu a(1-e^2)\}}\cos i, & h &= -\beta_3 = \Omega \end{aligned}$$

and the canonical equations become

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \quad \frac{dl}{dt} = -\frac{\partial R}{\partial L}$$

$$\frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}$$

$$\frac{dH}{dt} = \frac{\partial R}{\partial h}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}.$$

But there is here a change in the meaning of R due to replacing the element $-\beta$ by the *mean anomaly* l. If the disturbing function in the usual form quoted in § 139 be denoted by R_0, the variation of l follows from

$$\frac{d}{dt}(l - nt) = -\frac{\partial R_0}{\partial L}, \quad \frac{\partial R}{\partial L} = \frac{\partial R_0}{\partial L} - n$$

and therefore

$$R = R_0 - \int n\,dL = R_0 - \int \mu^2 L^{-3}dL = R_0 + \mu^2/2L^2.$$

This change in R has no effect in the other equations, and since R is a function of $\epsilon - \varpi + nt$, $\partial R/\partial l$ is the same thing as $-\partial R/\partial\beta$. The above canonical equations are precisely those on which Delaunay's theory of the Moon is based.

Without changing L let the transformation

$$L - G = \rho_1, \quad G - H = \rho_2, \quad -g - h = \omega_1, \quad -h = \omega_2, \quad l + g + h = \lambda$$

be made. Then

$$\lambda dL + \omega_1 d\rho_1 + \omega_2 d\rho_2 - (l dL + g dG + h dH) = 0$$

and this expression is therefore a perfect differential. Hence by § 125 the transformation from the variables

$$L, G, H; \ l, g, h$$

to the variables

$$L, \rho_1, \rho_2; \ \lambda, \omega_1, \omega_2$$

is one which leaves the equations of motion in the canonical form. The angle $\lambda = \epsilon + nt$ is the *mean longitude,* and $\omega_1 = -\varpi$, $\omega_2 = -\Omega$ are the longitudes of perihelion and the node, reversed in sign.

Again, consider the transformation

$$\xi = (2\rho)^{\frac{1}{2}} \cos \omega, \quad \eta = (2\rho)^{\frac{1}{2}} \sin \omega.$$

In this case

$$\eta d\xi - \omega d\rho = -2\rho \sin^2 \omega d\omega + \sin \omega \cos \omega d\rho - \omega d\rho$$
$$= d \{\rho (\tfrac{1}{2} \sin 2\omega - \omega)\}$$

is a perfect differential. Hence the variables $L, \rho_1, \rho_2; \lambda, \omega_1, \omega_2$ can be changed to

$$L, \xi_1, \xi_2; \ \lambda, \eta_1, \eta_2$$

and the canonical form of the equations will still be preserved. These variables have been used extensively by Poincaré. Since

$$\rho_1 = L - G = 2\sqrt{(\mu a)} \sin^2 \tfrac{1}{2}\phi$$

$(\sin \phi = e)$, ξ_1, η_1 are of the order of the eccentricity, and are called by him the *eccentric variables.* Similarly, since

$$\rho_2 = G - H = 2\sqrt{(\mu p)} \sin^2 \tfrac{1}{2} i$$

ξ_2, η_2 are of the same order as the inclination, and are therefore called the *oblique variables.*

144. The account which will be given of the lunar theory in later chapters will be based on a method which is quite different from Delaunay's. But the latter is in reality very general and therefore Delaunay's mode of integrating the canonical equations of the previous section will now be indicated. The form of the disturbing function will be taken to be

$$R = -B - A \cos (i_1 l + i_2 g + i_3 h + i_4 n't + q) + R_1$$
$$= -B - A \cos \theta + R_1 = R_0 + R_1$$

where R_1 represents an aggregate of periodic terms similar to the one written down and n', q are constants. The term B and the coefficients A are functions of L, G, H only and in comparison with B these coefficients are small quantities of definite orders. Let

$$\theta_1 = i_1 l + i_2 g + i_3 h = \theta - i_4 n't - q.$$

Then the variables

$$L, G, H; l, g, h$$

can be replaced by

$$L, G', H'; i_1^{-1}\theta_1, g, h$$

provided

$$(i_1^{-1}\theta_1 - l)\, dL + g\,.\,d(G' - G) + h\,.\,d(H' - H) = dW$$

is a perfect differential; and this condition is clearly satisfied if

$$G' = G - i_1^{-1}i_2L, \quad H' = H - i_1^{-1}i_3L$$

for then $dW = 0$. If now $R_1 = 0$, a solution of the problem can be found. For corresponding to the equation

$$R = -B - A\cos(\theta_1 + i_4n't + q)$$

the Hamilton-Jacobi equation takes the form

$$-B - A\cos\left(i_1\frac{\partial W}{\partial L} + i_4n't + q\right) + \frac{\partial W}{\partial t} = 0$$

and a solution involving three constants C, g', h' is

$$W = Ct + i_1^{-1}\int\theta\, dL - i_1^{-1}L(i_4n't + q) + g'G' + h'H'$$

provided

$$-B - A\cos\theta + C - i_1^{-1}L\,.\,i_4n' = 0.$$

This equation, which is in fact one integral, may be written

$$C = B_1 + A\cos\theta, \quad B_1 = B + i_4n'\,.\,i_1^{-1}L.$$

The solution, by § 126, takes the form ($\alpha_r = C, g', h'$; $\beta_r = c, -G', -H'$)

$$t + c + i_1^{-1}\frac{\partial}{\partial C}\int\theta\, dL = 0, \quad i_1^{-1}\theta_1 = i_1^{-1}(\theta - i_4n't - q)$$

$$G' = \text{const.}, \qquad g = g' + i_1^{-1}\frac{\partial}{\partial G'}\int\theta\, dL$$

$$H' = \text{const.}, \qquad h = h' + i_1^{-1}\frac{\partial}{\partial H'}\int\theta\, dL.$$

The lower limit of the integral involved is a function of C, G', H', but the integral is so defined that the integrand θ vanishes at this limit. The solution can also be written

$$L = i_1\Theta, \quad G = i_2\Theta + G', \quad H = i_3\Theta + H'$$

$$C = B_1 + A\cos\theta, \qquad B_1 = B + i_4n'\Theta$$

$$t + c = -\int\frac{\partial\theta}{\partial C}\, d\Theta = \int\frac{d\Theta}{\sqrt{\{A^2 - (C - B_1)^2\}}}$$

$$g = g' + \int\frac{\partial\theta}{\partial G'}\, d\Theta, \quad h = h' + \int\frac{\partial\theta}{\partial H'}\, d\Theta.$$

At this point $(C, g', h'; c, -G', -H')$ are absolute constants, resulting from the solution of a Hamilton-Jacobi equation when the Hamiltonian function is $R - R_1$. Hence, by § 127, the further treatment of the problem depends on taking these constants as new variables, and solving the canonical system

$$\frac{dC}{dt} = \frac{\partial R_1}{\partial c}, \quad \frac{dG'}{dt} = \frac{\partial R_1}{\partial g'}, \quad \frac{dH'}{dt} = \frac{\partial R_1}{\partial h'}.$$

$$\frac{dc}{dt} = -\frac{\partial R_1}{\partial C}, \quad \frac{dg'}{dt} = -\frac{\partial R_1}{\partial G'}, \quad \frac{dh'}{dt} = -\frac{\partial R_1}{\partial H'}.$$

But circumstances now arise which require further examination. For R_1 is now a function of the new variables, instead of the old, and the form of the function is important.

145. In the partial solution

$$C = B_1 + A\cos\theta, \quad \frac{d\Theta}{dt} = \sqrt{\{A^2 - (C - B_1)^2\}} = A\sin\theta$$

where B_1, A are functions of Θ (and the constants C, G', H'), and Θ, θ are functions of t to be determined. The forms to be expected may be seen in this way. The above equations give

$$\Theta = f(\cos\theta), \quad -f'(\cos\theta)\frac{d\theta}{dt} = A$$

and therefore

$$t + c = \int \phi(\cos\theta)\, d\theta = \theta/\theta_0 + \Sigma t_r \sin r\theta$$

when θ vanishes with $t + c$. Hence $\theta - \theta_0(t + c)$ is an odd periodic function of θ and therefore of $\lambda = \theta_0(t + c)$. Thus, θ_0 being some constant,

$$\theta = \lambda + \Sigma\theta_r \sin r\lambda, \quad \lambda = \theta_0(t + c)$$

and

$$\Theta = f(\cos\theta) = \Theta_0 + \Sigma\Theta_r \cos r\lambda.$$

These forms, which without a critical examination of the conditions have only been made plausible, are actually found in practice. It follows that

$L = i_1\Theta_0 + i_1\Sigma\Theta_r \cos r\lambda$, $G = G' + i_2\Theta_0 + i_2\Sigma\Theta_r\cos r\lambda$, $H = H' + i_3\Theta_0 + i_3\Sigma\Theta_r\cos r\lambda$

$$g = g' + \int \frac{\partial\theta}{\partial G'} \cdot \frac{A\sin\theta}{\theta_0} d\lambda = g' + g_0(t + c) + \Sigma g_r \sin r\lambda$$

$$h = h' + \int \frac{\partial\theta}{\partial H'} \cdot \frac{A\sin\theta}{\theta_0} d\lambda = h' + h_0(t + c) + \Sigma h_r \sin r\lambda$$

and the original variable l is given by

$$i_1 l = \theta - i_4 n't - q - i_2 g - i_3 h$$

$$= \lambda - i_4 n't - q - i_2\{g' + g_0(t+c)\} - i_3\{h' + h_0(t+c)\} + \Sigma(\theta_r - i_2 g_r - i_3 h_r)\sin r\lambda.$$

Now, since θ and Θ contain C, G', H', these constants also enter into g_0, h_0 and therefore into the coefficients of t in the arguments of the terms in R_1. Hence t will appear outside the circular functions in the derivatives of R_1 with respect to C, G', H'. This inconvenient circumstance must be avoided by a change of variables. Now

$$d\int \theta\, d\Theta = \theta d\Theta - (t+c)\, dC + (g-g')\, dG' + (h-h')\, dH'$$

by the form of the partial solution, and therefore

$$d\left(Ct - \int \Theta\, d\theta\right) = -\Theta d\theta - c dC + (g-g')\, dG' + (h-h')\, dH' + Cdt.$$

This is a perfect differential and when each side is expanded in the form of a secular and a periodic part, the same must clearly hold true for each part separately, at least when the number of periodic terms is finite; and in practice the remainder after a certain number of terms must be treated as negligible. But

$$\Theta \frac{d\theta}{d\lambda} = (\Theta_0 + \Sigma \Theta_r \cos r\lambda)(1 + \Sigma r\theta_r \cos r\lambda)$$

$$= \Lambda_0 + \Sigma \Lambda_r \cos r\lambda, \quad \Lambda_0 = \Theta_0 + \tfrac{1}{2}\Sigma r \Theta_r \theta_r.$$

Hence, when the periodic terms are omitted,

$$Cdt - \Lambda_0 d\lambda - cdC + g_0(t+c)\, dG' + h_0(t+c)\, dH'$$

is a perfect differential, to which $d(\Lambda_0\lambda)$ may be added; and therefore the variables

$$C,\ G',\ H';\ c,\ g',\ h'$$

can be replaced by

$$\Lambda_0,\ G',\ H';\ \lambda,\ \kappa,\ \eta$$

where

$$\kappa = g' + g_0(t+c), \quad \eta = h' + h_0(t+c).$$

This follows from § 125, which shows that at the same time R_1 must be replaced by $R_1 - C$. All is now expressed in terms of the last set of variables, and secular terms are thus removed from the arguments of the terms in R_1.

It is convenient to make a final simple transformation. Since

$$(i_1\lambda' - \lambda)\, d\Lambda_0 + i_2\kappa d\Lambda_0 + i_3\eta d\Lambda_0 = -d\{\Lambda_0(i_4 n't + q)\} + i_4 n'\Lambda_0 dt$$

if

$$i_1\lambda' = \lambda - i_2\kappa - i_3\eta - i_4 n't - q$$

the variables

$$\Lambda_0,\ G',\ H';\ \lambda,\ \kappa,\ \eta$$

can be replaced by

$$\Lambda' = i_1\Lambda_0,\ G'' = G' + i_2\Lambda_0,\ H'' = H' + i_3\Lambda_0;\ \lambda',\ \kappa,\ \eta$$

but at the same time it is necessary to add $i_4 n'\Lambda_0$ to $R_1 - C$. Thus finally, if

$$R' = R_1 - C + i_4 n'\Lambda_0$$

the system of canonical equations

$$\frac{d\Lambda'}{dt} = \frac{\partial R'}{\partial \lambda'}, \quad \frac{dG''}{dt} = \frac{\partial R'}{\partial \kappa}, \quad \frac{dH''}{dt} = \frac{\partial R'}{\partial \eta}$$

$$\frac{d\lambda'}{dt} = -\frac{\partial R'}{\partial \Lambda'}, \quad \frac{d\kappa}{dt} = -\frac{\partial R'}{\partial G''}, \quad \frac{d\eta}{dt} = -\frac{\partial R'}{\partial H''}$$

is obtained.

146. If the value of λ' be compared with the expression for l in terms of λ it will now be seen that

$$i_1 l = i_1\lambda' + \Sigma\,(\theta_r - i_2 g_r - i_3 h_r) \sin r\lambda$$

and thus λ' and l differ only by periodic terms. The same is true of κ, g and η, h. The periodic terms would disappear with A, as also those in Θ and θ, and Λ_0 would coincide with Θ_0 and Θ. Hence the final variables are the same as the original variables when $A = 0$. The form of R' differs from that of R mainly in the complete removal of the term $A \cos \theta$, and naturally the most important term will be first selected for elimination. Periodic terms will be introduced into the arguments of R', but it is easily seen that on expansion they give rise to periodic terms of a higher order than $A \cos \theta$.

The same process can be repeated indefinitely, until all sensible terms are one by one removed, together with those of a higher order introduced at an earlier stage. It has been assumed that i_1 is not zero. If $i_1 = 0$, $i_2 g$ or $i_3 h$ can take the place of $i_1 l$. There are also terms for which $i_1 = i_2 = i_3 = 0$. In the lunar problem these depend on the mean longitude of the Sun and are removed by a single preliminary operation analogous to the above.

Delaunay's expression for the disturbing function contains over 300 periodic terms, and their removal involves practically 500 operations of the above kind, reduced to the application of a set of formal rules. This immensely laborious task was carried out unaided. But the result is the most perfect analytical solution which has yet been found for the satellite type of motion in the problem of three bodies. The solution is not limited to the actual case of the Moon, since it is expressed in general algebraic terms. The satellite type of motion may indeed be defined as that type for which the Delaunay expansions are valid. It seems an interesting problem of the future whether such satellites as Jupiter VIII and IX will be found to satisfy this definition. Their conditions differ widely from those of the lunar problem, in particular in the fact that the motions are retrograde.

CHAPTER XIV

THE DISTURBING FUNCTION

147. The development of the disturbing function R in a suitable form gives rise to many difficulties, partly of analysis, partly of practical computation, and is the subject of an extensive literature*. It is possible to deal here only with a few of the more important points.

The principal part of the disturbing function for two planets involves the expansion of Δ^{-1}, the reciprocal of their mutual distance. It is therefore important to consider the nature of this expansion, or rather of Δ^{-2s} in general, where s is half an odd integer. For this more general form will give the derivatives of Δ^{-1}, Δ^2 being a rational quantity, and these will naturally occur when Δ^{-1} is expanded in terms of any contained parameter.

It is convenient to consider first the case of two circular, coplanar orbits. Then, if H is the difference of longitude in the plane,

$$\Delta^2 = a_1{}^2 + a_2{}^2 - 2a_1 a_2 \cos H$$

a_1, a_2 being the radii of the orbits. Let

$$a_1 < a_2, \quad \alpha = a_1/a_2, \quad \iota H = \log z, \quad \iota^2 = -1$$

and therefore

$$a_2{}^{-2}\Delta^2 = 1 + \alpha^2 - 2\alpha \cos H = (1 - \alpha z)(1 - \alpha z^{-1}).$$

Hence the function to be examined is

$$F^{-s} = (1 - \alpha z)^{-s}(1 - \alpha z^{-1})^{-s} = \tfrac{1}{2}\sum_{-\infty}^{\infty} b_s{}^i z^i$$

$$= (1 + \alpha^2 - 2\alpha \cos H)^{-s} = \tfrac{1}{2} b_s{}^0 + \sum_{1}^{\infty} b_s{}^i \cos iH.$$

Since the function is unaltered when z and z^{-1} are interchanged, $b_s{}^{-i} = b_s{}^i$, and i may be treated as positive. The coefficients $b_s{}^i$ are called *Laplace's coefficients*. By Fourier's theorem,

$$\left.\begin{aligned} b_s{}^i &= \frac{1}{\pi\iota}\int (1 - \alpha z)^{-s}(1 - \alpha z^{-1})^{-s} z^{i-1}\,dz \\ &= \frac{2}{\pi}\int_0^{\pi} (1 + \alpha^2 - 2\alpha \cos t)^{-s} \cos it\,dt \end{aligned}\right\} \quad \text{.....................(1)}$$

* Cf. H. v. Zeipel, *Encykl. der Math. Wiss.*, VI, 2, pp. 560–665.

The first (complex) integral is due to Cauchy; the path of integration is taken round a circle of unit radius. By introducing the Weierstrassian elliptic function

$$\wp(u) = z - \tfrac{1}{3}(\alpha + \alpha^{-1})$$

Cauchy's integral clearly becomes an elliptic function, and Poincaré has shown how this function can be reduced to a calculable form. But another method will be followed here.

The coefficients b_s^i are easily developed as power series in α^2. For, with the use of gamma functions,

$$(1-\alpha z)^{-s}(1-\alpha z^{-1})^{-s} = \sum_p \frac{\Gamma(s+p)}{\Gamma(s)\,\Gamma(p+1)}\alpha^p z^p \,.\, \sum_q \frac{\Gamma(s+q)}{\Gamma(s)\,\Gamma(q+1)}\alpha^q z^{-q}$$

and therefore, when $p = q + i$,

$$\tfrac{1}{2}b_s^i = \sum_q \frac{\Gamma(s+q+i)\,\Gamma(s+q)}{[\Gamma(s)]^2\,\Gamma(q+i+1)\,\Gamma(q+1)}\alpha^{2q+i}$$

$$= \frac{\Gamma(s+i)}{\Gamma(s)\,\Gamma(i+1)}\alpha^i \,.\, \sum_q \frac{\Gamma(s+q)}{\Gamma(s)} \cdot \frac{\Gamma(s+i+q)}{\Gamma(s+i)} \cdot \frac{\Gamma(i+1)}{\Gamma(i+1+q)} \cdot \frac{\alpha^{2q}}{\Gamma(q+1)}.$$

But this can be recognized as a hypergeometric series, and when it is expressed in the ordinary notation,

$$b_s^i = 2\alpha^i F(s,\ s+i,\ i+1,\ \alpha^2)\frac{\Gamma(s+i)}{\Gamma(s)\,\Gamma(i+1)} \quad \text{..............(2)}$$

By the known properties of the hypergeometric series, this expansion is convergent when $\alpha < 1$. There are many equivalent forms, but (2) is enough for the present purpose.

148. Laplace's coefficients are subject to several formulae of recurrence, which facilitate their calculation. That such exist follows from the known relations between sets of three contiguous hypergeometric functions. Instead of finding them directly, a more general function

$$B_s^{i,j} = \alpha^i\left(\frac{d}{d\alpha^2}\right)^j(\alpha^{-i}b_s^i)$$

may be considered, for this reduces to b_s^i when $j = 0$. In the integral (1) write $z = \alpha\zeta$, and then

$$\pi\iota\alpha^{-i}b_s^i = \int(1-\alpha^2\zeta)^{-s}(1-\zeta^{-1})^{-s}\zeta^{i-1}d\zeta.$$

It follows that

$$\pi\iota\alpha^{-i}B_s^{i,j} = \frac{\Gamma(s+j)}{\Gamma(s)}\int(1-\alpha^2\zeta)^{-s-j}(1-\zeta^{-1})^{-s}\zeta^{i+j-1}d\zeta.$$

The equivalent forms

$$\pi\iota\alpha^{-i}B_s^{i,j} = \frac{\Gamma(s+j)}{\Gamma(s)}\int(1-\alpha^2\zeta)^{-s-j-1}(1-\zeta^{-1})^{-s}(\zeta^{i+j-1}-\alpha^2\zeta^{i+j})\,d\zeta$$

$$= \frac{\Gamma(s+j)}{\Gamma(s)}\int(1-\alpha^2\zeta)^{-s-j}(1-\zeta^{-1})^{-s-1}(\zeta^{i+j-1}-\zeta^{i+j-2})\,d\zeta$$

show at once that

$$(s+j)\,B_s^{i,j} = \alpha\,B_s^{i-1,j+1} - \alpha^2\,B_s^{i,j+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\alpha\,B_s^{i,j} = s\,B_{s+1}^{i+1,j-1} - s\alpha\,B_{s+1}^{i,j-1}$$

Again,

$$\frac{d}{d\zeta}[(1-\alpha^2\zeta)^{-s-j+1}(1-\zeta^{-1})^{-s+1}\,\zeta^{i+j}]$$

$$=(1-\alpha^2\zeta)^{-s-j}(1-\zeta^{-1})^{-s}\,\{(s-i-1)\alpha^2\zeta^{i+j}+(i+j+i\alpha^2)\,\zeta^{i+j-1}-(i+j+s-1)\,\zeta^{i+j-2}\}.$$

When these expressions are integrated along a path lying between the limits $1<|\zeta|<\alpha^{-2}$, where the functions are regular, the first integrand returns to its original value. Therefore

$$(i-s+1)\,\alpha\,B_s^{i+1,j} - (i+j+i\alpha^2)\,B_s^{i,j} + (i+j+s-1)\,\alpha\,B_s^{i-1,j} = 0 \ldots(4)$$

The identity

$$(1-\alpha^2\zeta)^{-s-j}(1-\zeta^{-1})^{-s}\,\zeta^{i+j-1}$$

$$=(1-\alpha^2\zeta)^{-s-j-1}(1-\zeta^{-1})^{-s-1}\,\{(1+\alpha^2)\,\zeta^{i+j-1}-\alpha^2\zeta^{i+j}-\zeta^{i+j-2}\}$$

gives similarly on integration

$$(s+j)\,B_s^{i,j} = s\,(1+\alpha^2)\,B_{s+1}^{i,j} - s\alpha\,B_{s+1}^{i+1,j} - s\alpha\,B_{s+1}^{i-1,j}$$

and after eliminating the last term by means of (4) with $s+1$ in the place of s,

$$(i+j+s)(j+s)\,B_s^{i,j} = s\,[s+(j+s)\,\alpha^2]\,B_{s+1}^{i,j} - s\,(j+2s)\,\alpha\,B_{s+1}^{i+1,j} \ldots(5)$$

When $j=0$, (4) and (5) give formulae which apply to Laplace's coefficients. Derivatives of the latter with respect to α can then be expressed as linear functions of $B_s^{i,j}$.

149. Newcomb's method of calculating the coefficients b_s^i, together with their derivatives in the form subsequently required, can now be explained. Let

$$2s=n,\quad \delta=\frac{d}{d\alpha^2},\quad D=\alpha\frac{d}{d\alpha}=2\alpha^2\,.\,\delta$$

and let

$$c_n^{i,j} = 2^j\,\alpha^{s+2j-\frac{1}{2}}\,B_s^{i,j} = 2^j\,\alpha^{\frac{1}{2}(n-1)+i+2j}\,\delta^j\,(\alpha^{-i}b_s^i).$$

This is not Newcomb's definition of $c_n^{i,j}$, but it is the equivalent. Thus

$$D\,c_n^{i,j} = \{\tfrac{1}{2}(n-1)+i+2j\}\,c_n^{i,j} + c_n^{i,j+1}$$

and therefore

$$D^{k+1}c_n^{i,j} = \{\tfrac{1}{2}(n-1)+i+2j\}\,D^k c_n^{i,j} + D^k c_n^{i,j+1} \ldots\ldots\ldots\ldots(6)$$

so that these derivatives of a higher order are easily deduced from those of the next lower order. Let

$$p_n^{i,j} = c_n^{i,j}/c_n^{i-1,j} = B_s^{i,j}/B_s^{i-1,j}$$

and then, by (4),

$$p_n^{i,j} = \frac{P_n^{i,j}}{1 - Q_n^{i,j} p_n^{1+i,j}} \quad \text{.........................(7)}$$

where

$$P_n^{i,j} = \frac{(i + j + \frac{1}{2}n - 1)\,\alpha}{i\,(1 + \alpha^2) + j}, \quad Q_n^{i,j} = \frac{(i - \frac{1}{2}n + 1)\,\alpha}{i\,(1 + \alpha^2) + j}.$$

The development is to be carried to a definite order fixed by $i = k$, say 11. In the first place $p_n^{k,j}$ is calculated for the required values of n, j by a direct method. Next $p_n^{k-1,j}, \ldots, p_n^{1,j}$ are deduced in succession by (7). For $i = 1$, $s = \frac{1}{2}$, the formula (3) becomes

$$(2j + 1)\,\alpha\, c_1^{1,j} = c_1^{0,j+1} - \alpha\, c_1^{1,j+1} = c_1^{0,j+1}\,(1 - \alpha\, p_1^{1,j+1})$$

or

$$c_1^{0,j+1} = \frac{(2j + 1)\,\alpha p_1^{1,j} c_1^{0,j}}{1 - \alpha p_1^{1,j+1}} \quad \text{.........................(8)}$$

The first coefficient $c_1^{0,0}$ is calculated directly. Then (8) gives $c_1^{0,j}$ $(j = 1, 2, \ldots)$ in succession. The formula (5), when $i = 0$, gives

$$(j + \tfrac{1}{2}n)^2\,\alpha\, c_n^{0,j} = \tfrac{1}{2}n\,[\tfrac{1}{2}n + (j + \tfrac{1}{2}n)\,\alpha^2]\, c_{n+2}^{0,j} - \tfrac{1}{2}n\,(j + n)\,\alpha\, c_{n+2}^{1,j}$$

or

$$c_{n+2}^{0,j} = \frac{(j + \frac{1}{2}n)^2\,\alpha\, c_n^{0,j}}{\frac{1}{2}n\,[\frac{1}{2}n + (j + \frac{1}{2}n)\,\alpha^2] - \frac{1}{2}n\,(j + n)\,\alpha\, p_{n+2}^{1,j}} \quad \text{............(9)}$$

whence $c_n^{0,j}$ $(n = 3, 5, \ldots)$ are found in succession. It only remains to form $c_n^{i,j} = p_n^{i,j} c_n^{i-1,j}$ $(i = 1, 2, \ldots)$ and the calculation is then complete. The successive derivatives are finally derived by the use of (6).

The employment of a chain of recurrence formulae in practical computations requires care, because they are apt to involve an accumulation of numerical error. It is the merit of Newcomb's method here described that it is not only simple but very accurate.

150. The quantities which must be calculated directly are $c_1^{0,0}$ and $p_n^{k,j}$, where $n = 1, 3, \ldots, j = 0, 1, 2, \ldots$, and k is the highest value of i to which the expansion is carried. Now

$$c_1^{0,0} = b_{\frac{1}{2}}^{0} = \frac{2}{\pi}\int_0^{\pi} (1 + \alpha^2 - 2\alpha \cos t)^{-\frac{1}{2}}\, dt$$

a complete elliptic integral which can be found in a great variety of ways. Newcomb commends for the purpose the arithmetic-geometric mean, which Han
follows from the identity

$$\int_0^{\frac{1}{2}\pi} (a_n^2 \cos^2\phi + b_n^2 \sin^2\phi)^{-\frac{1}{2}}\, d\phi = \int_0^{\frac{1}{2}\pi} (a_{n+1}^2 \cos^2\psi + b_{n+1}^2 \sin^2\psi)^{-\frac{1}{2}}\, d\psi$$

where

$$2a_{n+1} = a_n + b_n, \quad b_{n+1}^2 = a_n b_n.$$

This is obtained immediately by the transformation of Gauss

$$\sin\phi = \frac{2a_n \sin\psi}{(a_n+b_n)\cos^2\psi + 2a_n\sin^2\psi}$$

and can be extended indefinitely by successive steps. It is obvious that the sequences a_n, b_n have a common limit A and hence that the value of the integral is $\pi/2A$. In the present case

$$a_1 = 1-\alpha, \quad b_1 = 1+\alpha, \quad c_1^{0,0} = 2A^{-1}$$

and this indicates one way in which $c_1^{0,0}$ is easily obtained.

The calculation of $p_n^{k,j}$ is based on the hypergeometric series (2). It is clear that

$$\delta F(s, s+i, i+1, \alpha^2) = \frac{s(s+i)}{i+1} F(s+1, s+i+1, i+2, \alpha^2)$$

and therefore generally

$$\delta^j F(s, \ldots) = \frac{\Gamma(s+j)}{\Gamma(s)} \cdot \frac{\Gamma(s+i+j)}{\Gamma(s+i)} \cdot \frac{\Gamma(i+1)}{\Gamma(i+j+1)} F(s+j, \ldots).$$

Hence, by (2),

$$B_s^{i,j} = \frac{\Gamma(s+j)}{[\Gamma(s)]^2} \cdot \frac{\Gamma(s+i+j)}{\Gamma(i+j+1)} \cdot 2\alpha^i F(s+j, s+i+j, i+j+1, \alpha^2)$$

and therefore, since $n = 2s$,

$$p_n^{i,j} = \frac{B_s^{i,j}}{B_s^{i-1,j}} = \frac{\frac{1}{2}n+i+j-1}{i+j} \cdot \frac{F(\frac{1}{2}n+j, \frac{1}{2}n+i+j, i+j+1, \alpha^2)}{F(\frac{1}{2}n+j, \frac{1}{2}n+i+j-1, i+j, \alpha^2)} \alpha.$$

The quotient of the two hypergeometric series can be converted into a continued fraction by a known theorem* of Gauss, and as it converges rapidly a few terms suffice to give its value. By this method Newcomb determined the required values of $p_n^{k,j}$.

151. In order to obtain the desired form of the continued fraction it is not necessary to introduce the hypergeometric series. By (3) and the following equation,

$$p_n^{i+1,j} = \frac{B_s^{i+1,j}}{B_s^{i,j}} = \frac{\alpha B_s^{i+1,j+1} - B_s^{i,j+1}}{\alpha B_s^{i,j+1} - B_s^{i-1,j+1}},$$

$$p_{n-2}^{i,j+2} = \frac{B_{s-1}^{i,j+2}}{B_{s-1}^{i-1,j+2}} = \frac{B_s^{i+1,j+1} - \alpha B_s^{i,j+1}}{B_s^{i,j+1} - \alpha B_s^{i-1,j+1}}$$

and by (4),

$$(i-s+1)\alpha B_s^{i+1,j+1} - (i+j+1+i\alpha^2) B_s^{i,j+1} + (i+j+s)\alpha B_s^{i-1,j+1} = 0.$$

* Chrystal's *Algebra*, II, p. 495.

These are three linear equations in $B_s^{i+1, j+1}$, $B_s^{i, j+1}$, $B_s^{i-1, j+1}$, which can be eliminated. The result may be expressed in the form:

$$\begin{vmatrix} (i-s+1)\alpha & i+j+i\alpha^2+1 & (i+j+s)\alpha \\ 1 & \alpha+p_{n-2}^{i,j+2} & \alpha p_{n-2}^{i,j+2} \\ \alpha & 1+\alpha p_n^{i+1,j} & p_n^{i+1,j} \end{vmatrix} = 0.$$

After expansion and division by $(1-\alpha^2)$ this gives

$$(i-s+1)\alpha p_n^{i+1,j} p_{n-2}^{i,j+2} - (i+j+1) p_n^{i+1,j} - i\alpha^2 p_{n-2}^{i,j+2} + (i+j+s)\alpha = 0$$

or

$$\{(i-s+1) p_n^{i+1,j} - i\alpha\}\{(i-s+1)\alpha p_{n-2}^{i,j+2} - (i+j+1)\} + (s+j)(1-s)\alpha = 0.$$

Therefore (7) gives $(2s = n)$

$$p_n^{i,j} = \frac{(i+j+s-1)\alpha}{i+j+i\alpha^2 - (i-s+1)\alpha p_n^{i+1,j}}$$

$$= \frac{(i+j+s-1)\alpha}{i+j-(s+j)(1-s)\alpha^2\{i+j+1-(i-s+1)\alpha p_{n-2}^{i,j+2}\}^{-1}}$$

$$= \frac{\frac{(i+j+s-1)\alpha}{i+j}}{1-} \; \frac{\frac{(s+j)(1-s)\alpha^2}{(i+j)(i+j+1)}}{1-} \; \frac{(i-s+1)\alpha p_{n-2}^{i,j+2}}{i+j+1}$$

$$= \frac{\frac{(i+j+s-1)\alpha}{i+j}}{1-} \; \frac{\frac{(s+j)(1-s)\alpha^2}{(i+j)(i+j+1)}}{1-} \; \frac{\frac{(i-s+1)(i+j+s)\alpha^2}{(i+j+1)(i+j+2)}}{1-} \cdots$$

and this is the required form. The relation between the alternate constituents is obvious enough, for the substitution of $j+2$ for j and $n-2$ for n (or $s-1$ for s) clearly has the effect of increasing each factor by 1 in the numerators and by 2 in the denominators. As $i = k$ is a fairly large number in the direct calculation of $p_n^{i,j}$, the even constituents are small and the calculation is based on an odd number of terms (generally five). With the use of subtraction logarithms the process is rapid.

152. The next step is to consider two circular orbits in planes inclined at an angle J. Let L_1, L_2 be the longitudes in the two planes, reckoned from the common node, and let

$$\mu = \cos^2 \tfrac{1}{2}J, \quad \nu = \sin^2 \tfrac{1}{2}J, \quad \mu + \nu = 1$$
$$x = L_1 - L_2, \quad y = L_1 + L_2.$$

Then the angular distance between the planets is given by

$$\cos H = \cos L_1 \cos L_2 + \sin L_1 \sin L_2 \cos J$$
$$= \mu \cos x + \nu \cos y$$

and

$$a_2\Delta^{-1} = (1+\alpha^2 - 2\alpha\cos H)^{-\frac{1}{2}}$$

$$= b^{0,0} + 2\sum_{i=1} b^{i,0}\cos ix + 2\sum_{j=1} b^{0,j}\cos jy + 4\sum_{i=1}\sum_{j=1} b^{i,j}\cos ix\cos jy$$

where

$$b^{i,j} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi (a_2\Delta^{-1})\cos ix\cos jy\,dx\,dy.$$

When ν is small Δ^{-1} can be expanded in powers of ν. Thus

$$a_2\Delta^{-1} = \{1+\alpha^2 - 2\alpha\cos x - 2\alpha\nu(\cos y - \cos x)\}^{-\frac{1}{2}}$$

$$= \sum_{n=0}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2})}(2\alpha\nu)^n(\cos y - \cos x)^n(1+\alpha^2-2\alpha\cos x)^{-n-\frac{1}{2}}\ldots(10)$$

or

$$2\sum_{i,j} b^{i,j}\xi^i\eta^j = \sum_n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2})}(\alpha\nu)^n(\eta+\eta^{-1}-\xi-\xi^{-1})^n\sum_i b^i_{n+\frac{1}{2}}\xi^i$$

where

$$\iota x = \log\xi, \quad \iota y = \log\eta, \quad \iota^2 = -1.$$

It is only necessary to compare the coefficients of $\xi^i\eta^j$ in these expressions in order to have $b^{i,j}$ as a power series in ν, the coefficients being functions of α. Thus, for example, as far as ν^2,

$$2b^{i,0} = b_{\frac{1}{2}}{}^{i} - \tfrac{1}{2}\alpha\nu\,(b_{\frac{3}{2}}{}^{i+1} + b_{\frac{3}{2}}{}^{i-1}) + \tfrac{3}{8}\alpha^2\nu^2\,(b_{\frac{5}{2}}{}^{i+2} + 4b_{\frac{5}{2}}{}^{i} + b_{\frac{5}{2}}{}^{i-2}) - \ldots$$

$$2b^{i,1} = \qquad \tfrac{1}{2}\alpha\nu\; b_{\frac{3}{2}}{}^{i} \qquad - \tfrac{3}{4}\alpha^2\nu^2\,(b_{\frac{5}{2}}{}^{i+1} + b_{\frac{5}{2}}{}^{i-1}) + \ldots$$

$$2b^{i,2} = \qquad \tfrac{3}{8}\alpha^2\nu^2\; b_{\frac{5}{2}}{}^{i} - \ldots.$$

It is easy to continue these developments further, and this is the method used by Le Verrier and Newcomb. But its validity is limited. The binomial expansion (10) of $a_2\Delta^{-1}$ is convergent only when

$$\nu < \left|\frac{1+\alpha^2-2\alpha\cos x}{2\alpha(\cos y - \cos x)}\right|$$

and since the most unfavourable case, $\cos x = -\cos y = 1$, must be included

$$\sin^2\tfrac{1}{2}J = \nu < (1-\alpha)^2/4\alpha.$$

It has been proved by H. v. Zeipel that the same limit applies to the expansion of *Jacobi's coefficients* $b^{i,j}$. This condition is satisfied in all cases by the small inclinations of the orbital planes of the major planets.

153. Among the orbits of the minor planets, however, are some whose inclinations to the plane of Jupiter exceed the above limit. It is therefore desirable to find a more general form of development. Let

$$F^{-s} = (1+\alpha^2 - 2\alpha\sigma)^{-s} = \Sigma C_s{}^n\alpha^n.$$

The coefficients C_s^n are polynomials in σ, which are in fact Legendre's polynomials when $s=\frac{1}{2}$. Differentiation with respect to σ and $\log\alpha$ gives

$$\frac{F^{s+2}}{2s\alpha}\Sigma\frac{dC_s^n}{d\sigma}\alpha^n=1+\alpha^2-2\alpha\sigma$$

$$\frac{F^{s+2}}{2s\alpha}\Sigma\frac{d^2C_s^n}{d\sigma^2}\alpha^n=2(s+1)\alpha$$

$$\frac{F^{s+2}}{2s\alpha}\Sigma\, nC_s^n\alpha^n=(\sigma-\alpha)(1+\alpha^2-2\alpha\sigma)$$

$$\begin{aligned}\frac{F^{s+2}}{2s\alpha}\Sigma\, n^2C_s^n\alpha^n&=(\sigma-2\alpha)(1+\alpha^2-2\alpha\sigma)+2(s+1)\alpha(\alpha-\sigma)^2\\&=(\sigma+2s\alpha)(1+\alpha^2-2\alpha\sigma)-2(s+1)\alpha(1-\sigma^2)\\&=\frac{F^{s+2}}{2s\alpha}\Sigma\left[-2snC_s^n+(2s+1)\sigma\frac{dC_s^n}{d\sigma}-(1-\sigma^2)\frac{d^2C_s^n}{d\sigma^2}\right]\alpha^n.\end{aligned}$$

Hence C_s^n satisfies the differential equation

$$(1-\sigma^2)\frac{d^2C}{d\sigma^2}-(2s+1)\sigma\frac{dC}{d\sigma}+n(n+2s)C=0\quad\text{............(11)}$$

Now in the present case

$$\sigma=\cos H=\mu\cos x+\nu\cos y$$

and the problem is to develop C_s^n in the form

$$C_s^n(\sigma)=\sum_{i,j}A^n_{i,j}\cos ix\cos jy\quad\text{.....................(12)}$$

where the coefficients $A^n_{i,j}$, considered generally as functions of μ, ν, are Appell's hypergeometric series in two variables μ^2, ν^2. But the solutions required can be deduced from the well known equation (11) by a certain treatment. It will be seen that this treatment is very special, but it is adequate for the purpose in view.

Let μ, ν, which are not in fact independent, for $\mu+\nu=1$, be considered as functions of a variable t. Their derivatives with respect to t will be denoted by μ', μ'', ν', ν''. Then

$$\frac{\partial^2C}{\partial x^2}=-\mu\cos x\frac{dC}{d\sigma}+\mu^2\sin^2x\frac{d^2C}{d\sigma^2}$$

$$\frac{\partial^2C}{\partial y^2}=-\nu\cos y\frac{dC}{d\sigma}+\nu^2\sin^2y\frac{d^2C}{d\sigma^2}$$

$$\frac{\partial C}{\partial t}=(\mu'\cos x+\nu'\cos y)\frac{dC}{d\sigma}$$

$$\frac{\partial^2C}{\partial t^2}=(\mu''\cos x+\nu''\cos y)\frac{dC}{d\sigma}+(\mu'\cos x+\nu'\cos y)^2\frac{d^2C}{d\sigma^2}.$$

It will now be seen that if with the help of these equations a partial differential equation can be deduced from (11), such that σ, $\cos x$ and $\cos y$

do not appear in it, a differential equation satisfied by $A^n_{i,j}$ will be deducible on comparing the coefficients of $\cos ix \cos jy$. Now

$$n(n+2s)C = (\mu^2\cos^2 x + \nu^2 \cos^2 y - 1 + 2\mu\nu \cos x \cos y)\frac{d^2C}{d\sigma^2}$$

$$+(2s+1)(\mu\cos x + \nu\cos y)\frac{dC}{d\sigma}$$

$$= \frac{\mu\nu}{\mu'\nu'}\frac{\partial^2 C}{\partial t^2} + \frac{d^2C}{d\sigma^2}\left[\mu^2\cos^2 x + \nu^2\cos^2 y - 1 - \frac{\mu\nu}{\mu'\nu'}(\mu'^2\cos^2 x + \nu'^2\cos^2 y)\right]$$

$$+\frac{dC}{d\sigma}\left[(2s+1)(\mu\cos x + \nu\cos y) - \frac{\mu\nu}{\mu'\nu'}(\mu''\cos x + \nu''\cos y)\right]$$

$$= \frac{\mu\nu}{\mu'\nu'}\frac{\partial^2 C}{\partial t^2} + \frac{d^2C}{d\sigma^2}\left[\mu^2 + \nu^2 - 1 - \frac{\mu\nu}{\mu'\nu'}(\mu'^2 + \nu'^2)\right]$$

$$+(\mu'\nu - \mu\nu')\left(\frac{1}{\mu\nu'}\frac{\partial^2 C}{\partial x^2} - \frac{1}{\mu'\nu}\frac{\partial^2 C}{\partial y^2}\right)$$

$$+\frac{dC}{d\sigma}\left[\left\{2s\mu - \frac{\nu}{\mu'\nu'}(\mu\mu'' - \mu'^2)\right\}\cos x + \left\{2s\nu - \frac{\mu}{\mu'\nu'}(\nu\nu'' - \nu'^2)\right\}\cos y\right]$$

and therefore if

$$M = \mu^2 + \nu^2 - 1 - \frac{\mu\nu}{\mu'\nu'}(\mu'^2 + \nu'^2) = 0$$

$$2s\frac{\mu}{\mu'} - \frac{\nu}{\nu'}\left(\frac{\mu\mu''}{\mu'^2} - 1\right) = 2s\frac{\nu}{\nu'} - \frac{\mu}{\mu'}\left(\frac{\nu\nu''}{\nu'^2} - 1\right) = N$$

the equation takes the required form

$$n(n+2s)C = \frac{\mu\nu}{\mu'\nu'}\frac{\partial^2 C}{\partial t^2} + (\mu'\nu - \mu\nu')\left(\frac{1}{\mu\nu'}\frac{\partial^2 C}{\partial x^2} - \frac{1}{\mu'\nu}\frac{\partial^2 C}{\partial y^2}\right) + N\frac{\partial C}{\partial t} \quad\text{......(13)}$$

154. At present μ and ν are any functions of t. Let

$$\mu^2 = (1-\rho_1)(1-\rho_2), \quad \nu^2 = \rho_1\rho_2.$$

Then it will easily be found that the first condition becomes

$$4\mu\mu'\nu\nu'M = (\rho_1 - \rho_2)^2\rho_1'\rho_2' = 0.$$

Hence either $\rho_1 = \rho_2$ or ρ_2 is independent of t. The first case has the more obvious importance since it gives directly

$$\nu = \rho_1 = \sin^2\tfrac{1}{2}J, \quad \mu = 1 - \rho_1 = \cos^2\tfrac{1}{2}J.$$

The second condition may be written

$$2s - 1 = \frac{\mu\nu}{\mu'\nu'}\cdot\frac{\mu''\nu' - \mu'\nu''}{\mu\nu' - \mu'\nu} \quad\text{........................(14)}$$

and the right-hand vanishes because $\mu + \nu = 1$. Hence the method can only be pursued further when $s = \frac{1}{2}$, but this happens to be the most important special case. If now $t = \nu$, $\nu' = -\mu' = 1$, $\mu'' = \nu'' = 0$, and the partial differential equation (13) in C becomes

$$n(n+1)C = -\nu(1-\nu)\frac{\partial^2 C}{\partial \nu^2} - \frac{1}{1-\nu}\frac{\partial^2 C}{\partial x^2} - \frac{1}{\nu}\frac{\partial^2 C}{\partial y^2} + (2\nu - 1)\frac{\partial C}{\partial \nu}.$$

On inserting the series (12) and comparing the coefficients of $\cos ix \cos jy$ this gives

$$n(n+1)A^n_{i,j} = -\nu(1-\nu)\frac{d^2A^n_{i,j}}{d\nu^2} + \left(\frac{i^2}{1-\nu} + \frac{j^2}{\nu}\right)A^n_{i,j} + (2\nu-1)\frac{dA^n_{i,j}}{d\nu}.$$

But the direct expansion of F^{-s} shows that since $\cos ix \cos jy$ arises from terms of the form $(\mu\cos x + \nu\cos y)^m$, $A^n_{i,j}$ must contain $\mu^i\nu^j$ as a factor. It is therefore proper to write

$$A^n_{i,j} = (1-\nu)^i \nu^j B^n_{i,j}$$

and this gives, with a little reduction,

$$n(n+1)B^n_{i,j} = (\nu^2-\nu)\frac{d^2B^n_{i,j}}{d\nu^2} + \{2\nu(i+j+1)-2j-1\}\frac{dB^n_{i,j}}{d\nu} + (i+j)(i+j+1)B^n_{i,j}$$

or

$$(\nu^2-\nu)\frac{d^2B^n_{i,j}}{d\nu^2} + \{2\nu(i+j+1)-2j-1\}\frac{dB^n_{i,j}}{d\nu} + (i+j-n)(i+j+1+n)B^n_{i,j} = 0.$$

Now $B^n_{i,j}$ is a polynomial in ν with a constant term, and this equation gives the law of its coefficients. But the equation is clearly of the form satisfied by a hypergeometric series. Hence

$$A^n_{i,j} = c\mu^i\nu^j F(i+j-n,\ i+j+1+n,\ 2j+1,\ \nu) \quad \text{.........(15)}$$

where c is a constant depending on i, j, n. This gives the form of Hansen's development in powers of α, namely

$$a_2\Delta^{-1} = \sum_{n,i,j} \alpha^n . A^n_{i,j} \cos ix \cos jy, \quad (n > i+j).$$

The determination of the constant c may be deferred.

155. This is the simplest, most obvious application of the method. But its possibilities, though limited, are not exhausted. The first condition for its use is also satisfied by making ρ_2 a constant. This may be expressed by

$$\rho_1 = \sin^2\tfrac{1}{2}J, \quad \rho_2 = \sin^2\tfrac{1}{2}J_0, \quad \mu = \cos\tfrac{1}{2}J\cos\tfrac{1}{2}J_0, \quad \nu = \sin\tfrac{1}{2}J\sin\tfrac{1}{2}J_0$$

where J_0 is to be treated initially as constant, though finally it will be identified with J. The relation $\mu+\nu=1$ no longer holds formally, but is replaced by

$$\mu^2/\cos^2\tfrac{1}{2}J_0 + \nu^2/\sin^2\tfrac{1}{2}J_0 = 1$$

and the result of differentiating this twice with respect to t and eliminating $\tan\tfrac{1}{2}J_0$ shows that the right-hand side of the second condition (14) is 1. Therefore $s=1$. At first sight this case has no present interest, since s is not half an odd integer, but the reason for considering it further will be seen later.

The development will be in powers of $\sin^2\tfrac{1}{2}J$ as before, but it will be convenient first to make $t=\tfrac{1}{2}J$, so that

$$\mu' = -\sin\tfrac{1}{2}J\cos\tfrac{1}{2}J_0, \quad \nu' = \cos\tfrac{1}{2}J\sin\tfrac{1}{2}J_0, \quad \mu'' = -\mu, \quad \nu'' = -\nu.$$

Then the partial differential equation (13) for C becomes

$$n(n+2)\,C = -\frac{\partial^2 C}{\partial t^2} - \sec^2 t\,\frac{\partial^2 C}{\partial x^2} - \operatorname{cosec}^2 t\,\frac{\partial^2 C}{\partial y^2} - 2\cot 2t\,\frac{\partial C}{\partial t}.$$

The form of the solution resembles the previous case, suggesting

$$C = \sum_{i,j} \mu^i \nu^j T^n_{i,j} \cos ix \cos jy$$

and the comparison of coefficients of $\cos ix \cos jy$ after the substitution gives

$$n(n+2)\,T^n_{i,j} = -\frac{d^2 T^n_{i,j}}{dt^2} - \{(2j+1)\cot t - (2i+1)\tan t\}\frac{dT^n_{i,j}}{dt} + (i+j)(i+j+2)\,T^n_{i,j}.$$

Now let the independent variable be changed to $\tau = \sin^2 t = \sin^2 \tfrac{1}{2}J$, so that

$$\frac{d}{dt} = 2\sin t\cos t\,\frac{d}{d\tau},\quad \frac{d^2}{dt^2} = 4\tau(1-\tau)\frac{d^2}{d\tau^2} + 2(1-2\tau)\frac{d}{d\tau}$$

and the previous equation becomes

$$4(\tau^2-\tau)\frac{d^2 T^n_{i,j}}{d\tau^2} + 4\{(i+j+2)\tau - (j+1)\}\frac{dT^n_{i,j}}{d\tau} + (i+j-n)(i+j+2+n)\,T^n_{i,j} = 0.$$

Now $T^n_{i,j}$ is a polynomial in τ with a constant term, and this equation determines the formation of its coefficients. But again it is an equation of the type satisfied by a hypergeometric series. Hence

$$T^n_{i,j} = c_1 F\left(\frac{i+j-n}{2}, \frac{i+j+2+n}{2}, j+1, \tau\right)$$

where c_1 is independent of τ. But μ and ν, and therefore $T^n_{i,j}$, involve J_0 symmetrically with J, and therefore it is evident that c_1 contains as a factor the same polynomial with τ replaced by $\tau_0 = \sin^2 \tfrac{1}{2}J_0$. Hence

$$T^n_{i,j} = c_2 F(\tau_0)\,F(\tau)$$

where c_2 is a constant independent of τ and τ_0. This is clearly general, whatever the values of J and J_0. A return to the actual problem can now be made by putting $J_0 = J$, and then $\tau = \nu$ and

$$T^n_{i,j} = c_2 F^2\left(\frac{i+j-n}{2}, \frac{i+j+2+n}{2}, j+1, \nu\right)$$

which gives the form of expansion

$$a_2^2\Delta^{-2} = \sum_{n,i,j} \alpha^n . T^n_{i,j}\,\mu^i \nu^j \cos ix \cos jy$$

$(i+j<n)$. The form of proof is essentially that of Stieltjes. The squared (terminating) hypergeometric series is a *polynomial of Tisserand.*

The more general utility of this result will now be easily seen. For

$$\begin{aligned} a_2^2\Delta^{-2} &= (1+\alpha^2 - 2\alpha\cos H)^{-1} = (1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1} \\ &= \{z(1-\alpha z)^{-1} - z^{-1}(1-\alpha z^{-1})^{-1}\}(z-z^{-1})^{-1} \\ &= \sum_n \alpha^n (z^{n+1} - z^{-n-1})(z-z^{-1})^{-1} \\ &= \sum_n \alpha^n . \sin(n+1)H/\sin H. \end{aligned}$$

Hence, by comparing the coefficients of α^n,

$$\sin(n+1)H/\sin H = \sum_{i,j} T^n_{i,j}\mu^i\nu^j \cos ix \cos jy.$$

But

$$(a_2^{-1}\Delta)^{-s} = \tfrac{1}{2}b_s^0 + \sum_1^\infty b_s^n \cos nH$$

$$= \tfrac{1}{2}b_s^0 + \Sigma\, \tfrac{1}{2}b_s^n \{\sin(n+1)H - \sin(n-1)H\}/\sin H$$

and therefore

$$(a_2^{-1}\Delta)^{-s} = \tfrac{1}{2}b_s^0 + \tfrac{1}{2}\sum_{n=1}^\infty b_s^n \sum_{i,j}(T^n_{i,j} - T^{n-2}_{i,j})\,\mu^i\nu^j \cos ix \cos jy \quad \ldots(16)$$

which is Tisserand's development in a series of Laplace's coefficients.

156. To complete the result it is necessary to find the numerical factor c_2. Now the final term of $F(-\alpha, \beta, \gamma, x)$, α, β, γ being positive integers, is

$$\frac{(\alpha+\beta-1)!\,(\gamma-1)!}{(\alpha+\gamma-1)!\,(\beta-1)!}(-x)^\alpha.$$

Hence the term containing the highest power of ν in $T^n_{i,j}\mu^i\nu^j$ is

$$(-1)^i c_2 \left\{\frac{n!\,j!}{[\tfrac{1}{2}(j-i+n)]!\,[\tfrac{1}{2}(i+j+n)]!}\right\}^2 \nu^n.$$

But

$$a_2^2\Delta^{-2} = \{1+\alpha^2 - 2\alpha\cos x - 2\alpha\nu(\cos y - \cos x)\}^{-1}$$

$$= \Sigma\,(2\alpha\nu)^m(\cos y - \cos x)^m(1+\alpha^2-2\alpha\cos x)^{-m-1}$$

and the highest power of ν associated with α^n is given by the terms

$$(\cos y - \cos x)^n(2\nu)^n = (\eta + \eta^{-1} - \xi - \xi^{-1})^n\nu^n$$

$$= (\eta - \xi)^n(1 - \xi^{-1}\eta^{-1})^n\nu^n$$

$$= \sum_{m,k}\frac{(n!)^2}{m!\,(n-m)!\,k!\,(n-k)!}\eta^m(-\xi)^{n-m}(-\xi\eta)^{-k}\nu^n$$

$$= \sum_{i,j}\frac{(-1)^i(n!)^2\xi^i\eta^j\nu^n}{[\tfrac{1}{2}(j-i+n)]!\,[\tfrac{1}{2}(i-j+n)]!\,[\tfrac{1}{2}(n-i-j)]!\,[\tfrac{1}{2}(n+i+j)]!}$$

when

$$m = \tfrac{1}{2}(j-i+n), \quad k = \tfrac{1}{2}(n-i-j).$$

The same terms appear in the form

$$\sum_{i,j} T^n_{i,j}\mu^i\nu^j \cos ix \cos jy = \kappa\sum_{i,j} T^n_{i,j}\mu^i\nu^j\xi^i\eta^j$$

where $\kappa = 1$ when i *and* $j = 0$, $\kappa = \tfrac{1}{2}$ when i *or* $j = 0$, and $\kappa = \tfrac{1}{4}$ otherwise. The highest power of ν has already been found in this form, and comparison of the coefficients of $\nu^n\xi^i\eta^j$ gives finally

$$c_2 = \kappa^{-1}\frac{[\tfrac{1}{2}(n+i+j)]!\,[\tfrac{1}{2}(n-i+j)]!}{(j!)^2\,[\tfrac{1}{2}(n+i-j)]!\,[\tfrac{1}{2}(n-i-j)]!}.$$

The development (16) is now completely defined.

The numerical factor c in Hansen's development (15) can be found similarly. For the term containing the highest power of ν in $A^n_{i,j}$ is

$$(-1)^{n-j} c \frac{(2n)!\,(2j)!}{(n+j-i)!\,(n+i+j)!} \nu^n.$$

On the other hand the terms associated with α^n and the highest power of ν in $a_2\Delta^{-1}$ are by (10) contained in

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2})}(\cos y - \cos x)^n (2\nu)^n$$

and these are now known. As before, the coefficients of $\nu^n \xi^i \eta^j$ in the two forms of $a_2\Delta^{-1}$ can be compared, and thus

$$(-1)^{n-j} \kappa c \frac{(2n)!\,(2j)!}{(n+j-i)!\,(n+i+j)!} = \frac{(-1)^i (n!)^2}{\Pi\{[\frac{1}{2}(n \pm i \pm j)]!\}} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2})}$$

where Π denotes the product of four factorial factors. Now $\frac{1}{2}(n-i-j)$ is an integer, $n-i-j$ is even, and the sign is the same on both sides. Also

$$\Gamma(n+1) = n!, \quad 2^{2n}\Gamma(n+\tfrac{1}{2}).\,n! = \Gamma(\tfrac{1}{2}).(2n)!.$$

Hence finally

$$c = \frac{(2^{2n}\kappa)^{-1}}{(2j)!} \frac{(n+i+j)!\,(n-i+j)!}{[\frac{1}{2}(n+i+j)]!\,[\frac{1}{2}(n-i+j)]!\,[\frac{1}{2}(n+i-j)]!\,[\frac{1}{2}(n-i-j)]!}$$

which completes the determination of Hansen's development.

The results obtained for inclined circular orbits may now be summarized. Since

$$\begin{aligned}\cos ix \cos jy &= \cos i(L_1 - L_2)\cos j(L_1 + L_2)\\ &= \tfrac{1}{2}\cos[(i+j)L_1 - (i-j)L_2] + \tfrac{1}{2}\cos[(i-j)L_1 - (i+j)L_2]\end{aligned}$$

it is possible to write

$$\Delta^{-1} = \Sigma A(p_1, p_2)\,\lambda_1^{p_1}\lambda_2^{p_2}, \quad 2i = |p_1 + p_2|, \quad 2j = |p_1 - p_2|$$

where $\log \lambda_1 = \iota L_1$, $\log \lambda_2 = \iota L_2$; and it has been shown how the coefficient $A(p_1, p_2)$ can be developed (1) in powers of $\nu = \sin^2 \frac{1}{2}J$, (2) in powers of $\alpha = a_1/a_2$, (3) as a series in Laplace's coefficients.

157. The preceding developments of Δ^{-1} or Δ^{-2s} apply to circular orbits, but they are not on that account to be regarded as mere approximations to the forms actually appropriate to the orbits of the solar system. On the contrary they constitute the essential source from which the latter forms must be generated by the most convenient means. Now quite generally

$$\Delta^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos H$$

and L_1, L_2 must be replaced by $\omega_1 + w_1$, $\omega_2 + w_2$, where ω_1, ω_2 are the longitudes of perihelion reckoned from the common node, and w_1, w_2 are the true anomalies. When the eccentricities e_1, e_2 vanish the radii r_1, r_2 become

the mean distances a_1, a_2, and w_1, w_2 can be identified with the mean anomalies M_1, M_2. The corresponding value of Δ may be written Δ_0.

Taylor's theorem can be expressed in the familiar symbolical form

$$f(x+y) = \exp.\left(y\frac{d}{dx}\right)f(x) = \exp.(yD)f(x)$$

which means simply that if the exponential function be expanded as though yD were an algebraic quantity, the result otherwise known to be true is formally reproduced. Thus generally,

$$f(x_1+y_1, x_2+y_2, \ldots) = \exp.(y_1D_1 + y_2D_2 + \ldots)f(x_1, x_2, \ldots)$$

where D_r operates on x_r alone. Now when $e_1 = e_2 = 0$,

$$\Delta_0^{-1} = f(a_1, a_2, L_1, L_2)$$

is an expansion of which the form has been completely determined. The more convenient developments refer not to $r-a$ but r/a, and the change from the argument a to the argument r is made additive by taking $\log a$ as the variable instead of a. Thus in the present case

$$x_1 = \log a_1, \quad x_2 = \log a_2, \quad x_3 = L_1 = \omega_1 + M_1, \quad x_4 = L_2 = \omega_2 + M_2$$

$$y_1 = \log r_1/a_1, \quad y_2 = \log r_2/a_2, \quad y_3 = w_1 - M_1, \quad y_4 = w_2 - M_2$$

$$D_1 = \frac{\partial}{\partial \log a_1} = a_1\frac{\partial}{\partial a_1}, \quad D_2 = \frac{\partial}{\partial \log a_2} = a_2\frac{\partial}{\partial a_2},$$

$$D_3 = \frac{\partial}{\partial L_1} = \iota\lambda_1\frac{\partial}{\partial \lambda_1}, \quad D_4 = \frac{\partial}{\partial L_2} = \iota\lambda_2\frac{\partial}{\partial \lambda_2}.$$

Then generally

$$\Delta^{-1} = F(r_1, r_2, w_1, w_2)$$

$$= \exp.\left[\log\frac{r_1}{a_1}.D_1 + \log\frac{r_2}{a_2}.D_2 + (w_1 - M_1)D_3 + (w_2 - M_2)D_4\right]f.$$

But in the notation of Hansen's coefficients (§ 45)

$$\left(\frac{r}{a}\right)^n x^m = \sum_i X_i^{n,m} z^i, \quad \left(\frac{r}{a}\right)^n\left(\frac{x}{z}\right)^m = \sum_i X_{i+m}^{n,m} z^i$$

where $\log x = \iota w$, $\log z = \iota M$. Hence in a corresponding symbolic notation, since $\log x/z = \iota(w-M)$,

$$\Delta^{-1} = \sum_i X_{i-\iota D_3}^{D_1, -\iota D_3} z_1{}^i . \sum_j X_{j-\iota D_4}^{D_2, -\iota D_4} z_2{}^j . f.$$

Simplifications are now possible owing to the form of f. In the first place Δ_0^{-1} is homogeneous, and of degree -1, in a_1, a_2. Hence

$$D_1 + D_2 = a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2} = -1.$$

But further f has been expanded in the form

$$f = \Sigma A(p_1, p_2) \lambda_1^{p_1} \lambda_2^{p_2}$$

and

$$D_3^q (\lambda_1^{p_1} \lambda_2^{p_2}) = (\iota p_1)^q \lambda_1^{p_1} \lambda_2^{p_2}, \quad D_4^q (\lambda_1^{p_1} \lambda_2^{p_2}) = (\iota p_2)^q \lambda_1^{p_1} \lambda_2^{p_2}$$

so that D_3, D_4 can be replaced by ιp_1, ιp_2, and D_1, D_2 do not operate on λ_1, λ_2. Hence the symbolic form of the complete expansion becomes

$$\Delta^{-1} = \sum_{p_1, p_2} \lambda_1^{p_1} \lambda_2^{p_2} \sum_{i,j} X_{i+p_1}^{D_1, p_1} X_{j+p_2}^{D_2, p_2} A(p_1, p_2) z_1^i z_2^j$$

where $\log \lambda_1 = \iota(\omega_1 + M_1)$, $\log \lambda_2 = \iota(\omega_2 + M_2)$, $\log z_1 = \iota M_1$, $\log z_2 = \iota M_2$, and the symbols X are respectively functions of e_1, D_1 and e_2, D_2.

158. This leads immediately to *Newcomb's operators* as defined by Poincaré. For the functions X can be expanded in positive powers of e, so that

$$X_{i+p_1}^{D_1, p_1} = \sum_{m_1} \Pi_i^{m_1}(D_1, p_1) e_1^{m_1}, \quad X_{j+p_2}^{D_2, p_2} = \sum_{m_2} \Pi_j^{m_2}(D_2, p_2) e_2^{m_2}$$

where $m_1 - |i|$, $m_2 - |j| = 0, 2, \ldots$, since $X_i^{n,m}$ is of the order $e^{|i-m|}$ at least. The operators Π are combined by Newcomb in the notation

$$\Pi_i^{m_1}(D_1, p_1) \Pi_j^{m_2}(D_2, p_2) = \Pi_{i,j}^{m_1, m_2} = \Pi_{i,0}^{m_1, 0} \Pi_{0,j}^{0, m_2}$$

but the combined symbols, though tabulated by him over a wide range, seem to present no practical advantage over the constituent operators.

The final form of the development of Δ^{-1} can therefore be written

$$\Delta^{-1} = \sum_{p_1, p_2} \lambda_1^{p_1} \lambda_2^{p_2} \sum_{m_1, m_2} e_1^{m_1} e_2^{m_2} \sum_{i,j} z_1^i z_2^j \Pi_i^{m_1}(D_1, p_1) \Pi_j^{m_2}(-1 - D_1, p_2) A(p_1, p_2)$$

and the completion of this part of the problem depends on the practical treatment of Newcomb's operators Π, which are polynomials in D, p of degree m, with numerical coefficients.

The definition of the symbols is given by

$$\sum_{m,i} \Pi_i^m(D, p) e^m z^i = \sum_i X_{i+p}^{D,p} z^i = \left(\frac{r}{a}\right)^D \left(\frac{x}{z}\right)^p.$$

Hence in particular

$$\sum_{m,i} \Pi_i^m(D, 0) e^m z^i = \left(\frac{r}{a}\right)^D, \quad \sum_{m,i} \Pi_i^m(0, p) e^m z^i = \left(\frac{x}{z}\right)^p$$

and therefore

$$\sum_{m,i} \Pi_i^m(D, p) e^m z^i = \sum_{m,i} \Pi_i^m(D, 0) e^m z^i \,.\, \sum_{n,j} \Pi_j^n(0, p) e^n z^j.$$

Comparison of the coefficients of $e^m z^i$ on both sides then gives

$$\Pi_i^m(D, p) = \sum_{n,j} \Pi_j^n(D, 0) \Pi_{i-j}^{m-n}(0, p)$$

where $n = 0, 1, \ldots, m$, and j has all the values which make $n - |j|$ and $m - n - |i - j|$ positive integers (including 0). This formula, due in another

notation to Cowell, makes the calculation of $\Pi_i^m(D, p)$ depend on the expansion of r/a and x^p.

But these are known forms. The first is given by (22) in Chapter IV. Means of deriving the latter have been given in § 45. In fact

$$X_{i+p}^{0,p} = \sum_m \Pi_i^m(0, p)\, e^m$$

and therefore it is necessary to expand $X_{i+p}^{0,p}$ in powers of e and the resulting coefficients will represent $\Pi_i^m(0, p)$. They are purely numerical and can be tabulated for all moderate values of m, i and p. Other methods have been suggested to facilitate the calculation of Newcomb's operators. But the above will suffice to make clear the principles involved.

159. The disturbing function due to the complete action of a single planet can now be considered. By (3) of § 23 this is

$$R = Gm'\left\{\frac{1}{\Delta} - \frac{1}{r'^3}(xx' + yy' + zz')\right\}$$

where (x, y, z), (x', y', z') are the heliocentric coordinates of the disturbed and disturbing planets; r' is the radius vector of the latter. The constant G may be reduced to unity by the choice of appropriate units, and the disturbing mass m' may be understood as a common factor to be restored ultimately. Thus

$$R = (r^2 + r'^2 - 2rr'\cos H)^{-\frac{1}{2}} - rr'^{-2}\cos H$$

where H has its previous meaning, the mutual elongation of the two planets as seen from the Sun. The principal part, already discussed, is symmetrical in r, r', but the indirect part is not so. Hence a distinction must be drawn, according as the disturbing planet is superior, when $r = r_1$, $r' = r_2$, or the disturbing planet is inferior, when $r = r_2$, $r' = r_1$. Now when the eccentricities vanish, by § 152,

$$a_2\Delta^{-1} = b^{0,0} + 2b^{1,0}\cos x + 2b^{0,1}\cos y + \ldots$$

$$\cos H = \qquad \mu\cos x + \qquad \nu\cos y$$

and

$$R - \Delta^{-1} = \delta R = -aa'^{-2}(\mu\cos x + \nu\cos y)$$

is the correction required to change Δ^{-1} into R. This can be effected by giving corrections to $b^{1,0}$ and $b^{0,1}$, thus

$$2\delta b^{1,0}/\mu = 2\delta b^{0,1}/\nu = -a_2 aa'^{-2}$$

$$= -\alpha\,(a' > a)\,;\ -\alpha^{-2}\,(a > a')$$

where $\alpha < 1$ always and a' is the mean distance of the disturbing planet. If these corrections are carried into the expansion in terms of ν (§ 152), as used in

the chief planetary theories, it will affect the Laplace's coefficients only to this extent:

$$\delta b_{\frac{1}{2}}^{\ 1} = -\alpha, \quad \delta b_{\frac{3}{2}}^{\ 0} = -2 \qquad (a' > a)$$

$$\delta b_{\frac{1}{2}}^{\ 1} = -\alpha^{-2}, \quad \delta b_{\frac{3}{2}}^{\ 0} = -2\alpha^{-3} \quad (a > a')$$

for it is easily verified that these changes will give the required corrections to $b^{1,0}$, $b^{0,1}$. In the exponential form they apply equally to $b^{-1,0}$, $b^{0,-1}$, and $b_{\frac{1}{2}}^{-1}$. Thus the indirect term is very simply incorporated in R_0, in which $e_1 = e_2 = 0$, and the full expansion of R in terms of the eccentricities can then be deduced in the manner explained for the development of Δ from Δ_0.

It is most important to remark that while the indirect part modifies the coefficients of certain elementary *periodic* terms, it affects in no way the *constant* term which is independent of the time.

160. Another order of development is possible by expanding Δ^{-1} initially in terms of r_1/r_2. If this ratio is small, as in the case of the solar perturbations of the lunar orbit, this method has great advantages. By § 153 this expansion takes the form

$$\Delta^{-1} = \sum_{n, i, j} r_1^{\,n} r_2^{-n-1} A^n_{i, j} \cos ix \cos jy$$

where $A^n_{i,j}$ is given by (15) and x, y have their true meanings,

$$W_1 \mp W_2 = \omega_1 + w_1 \mp (\omega_2 + w_2).$$

It is more convenient to use the exponential form, and with a slight change of notation for the coefficients,

$$\Delta^{-1} = \sum_{n, p_1, p_2} r_1^{\,n} r_2^{-n-1} A_n (p_1, p_2) \mu_1^{p_1} \mu_2^{p_2}$$

where $\log \mu_1 = \iota (\omega_1 + w_1)$, $\log \mu_2 = \iota (\omega_2 + w_2)$, $|p_1 - p_2| = 2i$, $|p_1 + p_2| = 2j$ and $n - |p_1|$, $n - |p_2|$ are even positive integers. Hence

$$\Delta^{-1} = \sum_{n, p_1, p_2} r_1^{\,n} r_2^{-n-1} A_n (p_1, p_2) \lambda_1^{p_1} \lambda_2^{p_2} (x_1 z_1^{-1})^{p_1} (x_2 z_2^{-1})^{p_2}$$

where $\log \lambda_1 = \iota (\omega_1 + M_1)$, $\log \lambda_2 = \iota (\omega_2 + M_2)$, $\log z_1 = \iota M_1$, $\log z_2 = \iota M_2$, $\log x_1 = \iota w_1$, $\log x_2 = \iota w_2$. But this form can clearly be expressed in terms of Hansen's coefficients. Thus

$$\Delta^{-1} = \sum_{n, p_1, p_2} \sum_{q_1, q_2} a_1^{\,n} a_2^{-n-1} A_n (p_1, p_2) \lambda_1^{p_1} \lambda_2^{p_2} X^{n,\, p_1}_{q_1 + p_1} X^{-n-1,\, p_2}_{q_2 + p_2} z_1^{q_1} z_2^{q_2}$$

where q_1, q_2 have all integral values, positive and negative, and the symbols X are respectively functions of e_1, e_2, while $A_n (p_1, p_2)$ is a function of $\nu = \sin^2 \frac{1}{2} J$ which has been determined.

The indirect part of the disturbing function, when r_1 $(< r_2)$ refers to the disturbed body, is clearly allowed for by simply excluding the terms corresponding to $n = 1$, for these are equal to $r_1 r_2^{-2} \cos H$.

By either method the fundamental importance of Hansen's coefficients and their relation to Newcomb's symbolic operators is clearly seen. Numerical developments of their coefficients according to powers of e have been calculated by several authors, including Cayley, Newcomb and, for the purposes of the lunar theory, Delaunay.

161. It has been seen that the generating expansion is of the form

$$\begin{aligned} R &= \Sigma\, 2A\mu^p\nu^q \cos px \cos qy \\ &= \Sigma\ \ A\mu^p\nu^q \cos\left[(p+q)L-(p-q)L'\right]\end{aligned}$$

where $L=\omega+M$, $L'=\omega'+M'$. The subsequent process introduces e, e' into the coefficient A, which already contains powers of $\nu=\sin^2\frac{1}{2}J$, and adds multiples of M, M' to the argument. In the ordinary notation for the elements,

$$\omega=\varpi-\Omega-\chi,\quad \omega'=\varpi'-\Omega'-\chi'$$

where χ, χ' are the distances of the intersection of the orbits from their ecliptic nodes. Hence R takes the form

$$\begin{aligned} R=\Sigma A\mu^p\nu^q \cos[hM+h'M'+(p+q)(\varpi-\Omega)& \\ -(p-q)(\varpi'-\Omega')-p(\chi-\chi')-q(\chi+\chi')]&.\end{aligned}$$

Now the two orbits with the ecliptic form a spherical triangle ABC in which

$$\begin{aligned} &a=\chi', \quad b=\chi, \quad c=\Omega_2-\Omega_1 \\ &A=i, \quad B=\pi-i', \quad C=J\end{aligned}$$

where i, i' are the inclinations of the orbits to the ecliptic. Hence, as in § 67, if the intersection be taken as the ascending node of the disturbing orbit on the disturbed orbit,

$$\begin{aligned} \sin\tfrac{1}{2}(\chi+\chi')\sin\tfrac{1}{2}J &= \sin\tfrac{1}{2}(\Omega'-\Omega)\sin\tfrac{1}{2}(i'+i) \\ \cos\tfrac{1}{2}(\chi+\chi')\sin\tfrac{1}{2}J &= \cos\tfrac{1}{2}(\Omega'-\Omega)\sin\tfrac{1}{2}(i'-i) \\ \sin\tfrac{1}{2}(\chi-\chi')\cos\tfrac{1}{2}J &= \sin\tfrac{1}{2}(\Omega'-\Omega)\cos\tfrac{1}{2}(i'+i) \\ \cos\tfrac{1}{2}(\chi-\chi')\cos\tfrac{1}{2}J &= \cos\tfrac{1}{2}(\Omega'-\Omega)\cos\tfrac{1}{2}(i'-i)\end{aligned}$$

and therefore

$$\nu^{\frac{1}{2}}\exp.\tfrac{1}{2}\iota(\chi+\chi')=\sin\tfrac{1}{2}i'\cos\tfrac{1}{2}i\exp.\tfrac{1}{2}\iota(\Omega'-\Omega)-\sin\tfrac{1}{2}i\cos\tfrac{1}{2}i'\exp.-\tfrac{1}{2}\iota(\Omega'-\Omega)$$

$$\mu^{\frac{1}{2}}\exp.\tfrac{1}{2}\iota(\chi-\chi')=\cos\tfrac{1}{2}i'\cos\tfrac{1}{2}i\exp.\tfrac{1}{2}\iota(\Omega'-\Omega)+\sin\tfrac{1}{2}i\sin\tfrac{1}{2}i'\exp.-\tfrac{1}{2}\iota(\Omega'-\Omega).$$

It follows that

$$\begin{aligned} &\nu^q\cos q(\chi+\chi')=\Sigma\, b_s\cos s(\Omega'-\Omega), \quad \nu^q\sin q(\chi+\chi')=\Sigma\, b_s\sin s(\Omega'-\Omega) \\ &\mu^p\cos p(\chi-\chi')=\Sigma\, a_s\cos s(\Omega'-\Omega), \quad \mu^p\sin p(\chi-\chi')=\Sigma\, a_s\sin s(\Omega'-\Omega)\end{aligned}$$

where a_s, b_s represent simple coefficients involving i, i'. Thus $\chi\pm\chi'$ can be eliminated from R, which now takes the form

$$R=\Sigma A\cos[hM+h'M'+(p+q)(\varpi-\Omega)-(p-q)(\varpi'-\Omega')-(s+s')(\Omega'-\Omega)]$$

where A now contains a, a', e, e', i, i' and also powers of ν. But from the above analogies of Delambre,

$$\nu = \sin^2 \tfrac{1}{2}(\Omega' - \Omega)\sin^2 \tfrac{1}{2}(i' + i) + \cos^2 \tfrac{1}{2}(\Omega' - \Omega)\sin^2 \tfrac{1}{2}(i' - i)$$
$$= \tfrac{1}{2}(1 - \cos i \cos i') - \tfrac{1}{2}\sin i \sin i' \cos(\Omega' - \Omega).$$

Hence these powers of ν can be removed from the coefficient without altering the form of the arguments, which are only changed by the addition of some multiples of $\Omega' - \Omega$. Thus finally

$$R = \Sigma A \cos[hM + h'M' + g\varpi + g'\varpi' + f\Omega + f'\Omega']$$
$$= \Sigma A \cos[h(nt + \epsilon) + h'(n't + \epsilon') + g\varpi + g'\varpi' + f\Omega + f'\Omega']$$

where the coefficient A is now a function of a, a', e, e', i, i' only, and the argument contains the six elements Ω, Ω', ϖ, ϖ', ϵ, ϵ' and the time. And this is the final form of the disturbing function, involving the twelve elements of the two orbits explicitly, and expressed in the desired way.

CHAPTER XV

ABSOLUTE PERTURBATIONS

162. The disturbance of a purely elliptic motion may be illustrated in a quite elementary way by supposing the motion to take place in a resisting medium. Let the tangential resistance per unit mass be $\alpha v/r^2$, where v is the velocity and r the radius vector, so that the radial and tangential components are

$$-\frac{\alpha v}{r^2}\cdot\frac{1}{v}\frac{dr}{dt}=-\frac{\alpha}{r^2}\frac{dr}{dt},\quad -\frac{\alpha v}{r^2}\cdot\frac{r}{v}\frac{d\theta}{dt}=-\frac{\alpha}{r}\frac{d\theta}{dt}.$$

When other powers of v and r are assumed in the expression for the resistance the general results are very much the same, and this simple form is sufficiently typical to represent fairly an interesting problem.

Let u be the reciprocal of r and δW the work done by external forces in a small radial or transversal displacement. Then

$$-u^2\frac{\partial W}{\partial u}=-\mu u^2+\alpha\frac{du}{dt},\quad u\frac{\partial W}{\partial\theta}=-\alpha u\frac{d\theta}{dt}$$

where μ is the constant of attraction; and the kinetic energy is T, where

$$2T=\dot{r}^2+r^2\dot{\theta}^2=u^{-4}\dot{u}^2+u^{-2}\dot{\theta}^2.$$

Hence the equations of motion are

$$\frac{d}{dt}(u^{-4}\dot{u})+2u^{-5}\dot{u}^2+u^{-3}\dot{\theta}^2=\mu-\alpha u^{-2}\frac{du}{dt}$$

$$\frac{d}{dt}(u^{-2}\dot{\theta})\qquad=\quad-\alpha\frac{d\theta}{dt}.$$

Now let

$$u^{-2}\dot{\theta}=H,\quad \frac{d}{dt}=Hu^2\frac{d}{d\theta}$$

and the first equation of motion becomes

$$Hu^2\frac{d}{d\theta}\left(Hu^{-2}\frac{du}{d\theta}\right)+2H^2u^{-1}\left(\frac{du}{d\theta}\right)^2+H^2u=\mu-\alpha H\frac{du}{d\theta}$$

or

$$H^2\left(\frac{d^2u}{d\theta^2}+u\right)+H\left(\frac{dH}{d\theta}+\alpha\right)\frac{du}{d\theta}-\mu=0.$$

But by the second equation of motion

$$H = h - \alpha\theta$$

where h is constant. Hence

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{(h - \alpha\theta)^2} = 0.$$

It is enough to retain the first power of α, so that

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\left(1 + \frac{2\alpha\theta}{h}\right)$$

and the integral is

$$u = \mu h^{-2}\{1 + e\cos(\theta - \gamma) + 2\alpha h^{-1}\theta\} \text{......................} (1)$$

where e and γ are constants.

163. The *osculating ellipse* at the point $\theta = \theta_1$ is obtained by supposing the resisting medium to disappear at this point and the subsequent motion under the central attraction to be undisturbed. The path is then

$$u = p_1^{-1}\{1 + e_1\cos(\theta - \gamma_1)\}.$$

The motion at the instant is the same in the actual trajectory (1) and in this ellipse, and thus $\theta = \theta_1$, $u = u_1$, $\dot{u}$ and $\dot{\theta}$, and therefore $H = H_1$ and $du/d\theta$ are the same for both curves. Let $\mu h^{-2} = p^{-1}$. Now H_1 is the constant of areal velocity in the ellipse, and hence

$$p_1^{-1} = \mu H_1^{-2} = p^{-1}(1 - \alpha h^{-1}\theta_1)^{-2}.$$

To the first order in α then

$$p_1^{-1}\Delta p_1 = -2\alpha h^{-1}\theta_1.$$

Again, by equating the values of u and $du/d\theta$,

$$p_1^{-1}\{1 + e_1\cos(\theta_1 - \gamma_1)\} = p^{-1}\{1 + e\cos(\theta_1 - \gamma) + 2\alpha h^{-1}\theta_1\}$$
$$p_1^{-1}\{\ \ -e_1\sin(\theta_1 - \gamma_1)\} = p^{-1}\{\ \ -e\sin(\theta_1 - \gamma) + 2\alpha h^{-1}\}$$

and to the first order in α

$$e_1\cos(\theta_1 - \gamma_1) = e\cos(\theta_1 - \gamma) - 2\alpha h^{-1}e\theta_1\cos(\theta_1 - \gamma)$$
$$e_1\sin(\theta_1 - \gamma_1) = e\sin(\theta_1 - \gamma) - 2\alpha h^{-1} - 2\alpha h^{-1}e\theta_1\sin(\theta_1 - \gamma).$$

Hence

$$e_1\cos(\gamma_1 - \gamma) = e - 2\alpha h^{-1}e\theta_1 - 2\alpha h^{-1}\sin(\theta_1 - \gamma)$$
$$e_1\sin(\gamma_1 - \gamma) = 2\alpha h^{-1}e\cos(\theta_1 - \gamma)$$

and, still to the first order,

$$\Delta e_1 = -2\alpha h^{-1}\{e\theta_1 + \sin(\theta_1 - \gamma)\}$$
$$\Delta\gamma_1 = \ \ 2\alpha h^{-1}\cos(\theta_1 - \gamma).$$

Between these terms an important practical distinction is at once apparent. That in Δe_1 depending on θ_1 will diminish the eccentricity indefinitely until the orbit becomes circular. It is a *secular* term. The other terms are

periodic, and when α is small their effect, not being cumulative, is small also. In practical applications, to Encke's comet for example, they can be neglected. Then $\Delta\gamma_1 = 0$ and the direction of the apsidal line is unaffected by the resisting medium.

In a complete revolution the secular effects are given by

$$\frac{\Delta e_1}{e_1} = \frac{\Delta p_1}{p_1} = -\frac{4\pi\alpha}{h}$$

and the corresponding changes in the mean motion and the mean distance are given by

$$\frac{\Delta n_1}{n_1} = -\frac{3}{2}\frac{\Delta a_1}{a_1} = -\frac{3}{2}\frac{\Delta p_1}{p_1} - \frac{3e_1\Delta e_1}{1-e_1^2} = \frac{1+e_1^2}{1-e_1^2} \cdot \frac{6\pi\alpha}{h}$$

since $a_1 = p_1(1-e_1^2)^{-1}$. Thus the most important effects of a resisting medium are a steady increase in the mean motion and a steady decrease in the mean distance, which must ultimately bring the disturbed body into contact with the centre of attraction.

164. This simple example has been chosen, apart from its intrinsic interest, because it illustrates certain important points. There is, in the first place, the osculating or instantaneous ellipse, which is

$$p_1 u = 1 + e_1 \cos(\theta - \gamma_1)$$

and not

$$pu = 1 + e \cos(\theta - \gamma).$$

The latter is a definite curve which may be called an intermediate orbit and may serve usefully as a curve of reference. Indeed it has been so used in what precedes. But it is not the osculating orbit at any time. There is also the distinction drawn between periodic and secular disturbances in the motion, of which the former may be relatively unimportant compared with the latter because these, however slow, are cumulative in effect.

The general nature of disturbed planetary motion can now be considered. For two planets only, the disturbing function has the form, found in the last chapter,

$$R = \Sigma F(a, a', e, e', i, i') \cos T,$$

$$T = [h(nt+\epsilon) + h'(n't+\epsilon') + g\varpi + g'\varpi' + f\Omega + f'\Omega']$$

where $(a, n, e, i, \Omega, \varpi, \epsilon)$ are the elements of the disturbed orbit, $(a', n', e', i', \Omega', \varpi', \epsilon')$ the elements of the disturbing orbit. The equations of § 139 are now available for finding the variations of the elements. In accordance with the artifice explained in § 140 the mean longitude ϵ is taken in a special sense there defined, and a in the coefficient and n in the argument of any term are treated as independent in forming the partial differential coefficients of R. Therefore

$$\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}, \frac{\partial R}{\partial i}$$

are all of the form $\Sigma C \cos T$, and

$$\frac{\partial R}{\partial \Omega}, \quad \frac{\partial R}{\partial \varpi}, \quad \frac{\partial R}{\partial \epsilon}$$

are all of the form $\Sigma C \sin T$, where T is the argument of the term. Hence the equations for the variations are themselves of the form

$$\frac{da}{dt} = \Sigma C_1 \sin T, \ldots$$

$$\frac{d\Omega}{dt} = \Sigma C_2 \cos T, \ldots.$$

In the first approximation the right-hand members (which contain the disturbing mass as a factor) are calculated with the osculating elements of both orbits for a certain epoch, and these elements are treated as constant. The equations can then be integrated, and in fact

$$\delta_1 a = -\Sigma C_1 \cos T/(hn + h'n'), \ldots$$
$$\delta_1 \Omega = \quad \Sigma C_2 \sin T/(hn + h'n'), \ldots.$$

These are the *absolute perturbations* of the first order. Similarly the perturbations of the first order in the masses can be calculated for all the disturbing planets concerned and the results can be combined by addition.

165. Each term in the perturbations represents a distinct *inequality* in the motion of the disturbed planet. It will now be seen that the inequalities are of two kinds. The multipliers h, h' have all integral values, positive and negative, including 0. When $h = h' = 0$ the disturbing function R is reduced to that part which does not contain the time. Thus

$$\frac{da}{dt} = C_1, \ldots, \quad \frac{d\Omega}{dt} = C_2, \ldots$$

$$\delta_1 a = C_1 t, \ldots, \quad \delta_1 \Omega = C_2 t, \ldots$$

and the inequalities are *secular*. From the present limited point of view they will increase indefinitely and in the course of time will modify the conditions of the planetary system profoundly, uncompensated by any check.

But one remark can be made immediately. The most important element as regards the stability of the system is clearly the mean distance a. Now when $h = h' = 0$, not only does t disappear from R but also ϵ. Hence

$$\frac{da}{dt} = \frac{\partial R}{\partial \epsilon} \cdot 2\sqrt{\left(\frac{a}{\mu}\right)} = 0$$

and in the previous set of equations $C_1 = 0$. There is therefore no secular inequality in a of the first order in the masses. How far this important theorem can be extended to the higher orders must be seen later. It follows that the mean motion n is also free from any secular inequality of the first order.

The other inequalities, when h and h' are not both zero, are evidently purely periodic, unless $hn + h'n' = 0$. The meaning of this qualification is that the mean motions must not be commensurable. Now mean motions are never commensurable, except perhaps instantaneously, since in fact they are not constant. But there are, as it were, degrees of incommensurability. In any case integers can be found to make $hn + h'n'$ smaller than any assignable quantity. If the incommensurability of n, n' is high, the corresponding integers h, h' will be large. In general the coefficients in R which correspond to arguments of a high order diminish rapidly with the order. Then the occurrence of a small divisor $hn + h'n'$ on integration will have no very serious effect. But if the incommensurability of the mean motions is low, this divisor may become very small for quite moderate values of h, h', and a fairly small term in the disturbing function may be greatly magnified by integration.

Thus in the case of Jupiter and Saturn

$$5n - 2n' = n/30 = n'/74$$

nearly, and this fact causes a considerable inequality in the motion of both planets, with a period of nearly 900 years. The period of such an inequality is $2\pi/(hn + h'n')$ and therefore inequalities of the class just considered are always connected with long periods. They hold an intermediate place between ordinary periodic inequalities and secular inequalities.

The mean longitude is affected in a double degree. For (§ 140) this is

$$\epsilon + \int n\,dt = \epsilon + \rho$$

where

$$\frac{d^2\rho}{dt^2} = -\frac{3}{a^2}\frac{\partial R}{\partial \epsilon} = \Sigma\, C \sin T$$

and therefore

$$\delta_1\rho = -\Sigma\, C \sin T/(hn + h'n')^2.$$

The long-period inequalities in the other elements have the divisor $hn + h'n'$ in the first degree only. Hence the principal effect is to be observed in the mean longitude.

166. It is in the next place necessary to consider the perturbations of the second order in the masses, for the first approximation does not in general suffice, and in the theories of Jupiter and Saturn it is even necessary to go beyond the third order. It is convenient to write

$$a = a_0 + \delta_1 a_0 + \delta_2 a_0 + \ldots,\ \ldots,\ \epsilon = \epsilon_0 + \delta_1 \epsilon_0 + \delta_2 \epsilon_0 + \ldots$$

$$a' = a_0' + \delta_1 a_0' + \delta_2 a_0' + \ldots,\ \ldots,\ \epsilon' = \epsilon_0' + \delta_1 \epsilon_0' + \delta_2 \epsilon_0' + \ldots$$

where $a_0, \ldots, \epsilon_0, a_0', \ldots, \epsilon_0'$ are the osculating elements for a chosen epoch, and δ_1 indicates the perturbations of the first order, the derivation of which has been

explained, δ_2 those of the second order, and so on. The equations for the variations of the elements can be written, for example, in the form

$$\frac{d\Omega}{dt} = \frac{(\mu a)^{-\frac{1}{2}}}{\cos\phi \sin i} \frac{\partial R}{\partial i} = m' f(a, a', \ldots, \rho + \epsilon, \rho' + \epsilon')$$

and after substituting the above expressions for $a, \ldots, \epsilon'$ and expanding by Taylor's theorem,

$$\frac{d}{dt}(\delta_2 \Omega) = m' \left\{ \delta_1 a_0 \frac{\partial f}{\partial a_0} + \delta_1 a_0' \frac{\partial f}{\partial a_0'} + \ldots + (\delta_1 \rho_0 + \delta_1 \epsilon_0) \frac{\partial f}{\partial \epsilon_0} + (\delta_1 \rho_0' + \delta_1 \epsilon_0') \frac{\partial f}{\partial \epsilon_0'} \right\}.$$

The reduction of the right-hand side to a suitable form will be readily understood in general terms, apart from the complexities which will naturally arise in the practical calculation, and a simple integration, requiring the introduction of no arbitrary constant, will give the expression of $\delta_2 \Omega$. Similarly the perturbations of higher orders, so far as they are of sensible magnitude, can be found successively, when those of the lower orders have been determined, for all the elements.

167. The general form of the results will now be apparent. In the first order the inequalities are of the forms

$$A \cos(\nu t + h), \quad At$$

only. In the higher orders the terms obtained by the algebraic composition and subsequent integration of these two forms will clearly belong to one of the three types

$$A \cos(\nu t + h), \quad At^m, \quad At^m \cos(\nu t + h)$$

which may be called respectively periodic, purely secular and mixed terms. The term *order* may be retained to denote the degree α of A in the masses. As A is also a function of the eccentricities and inclinations, which are also in general small parameters, it may be limited to a homogeneous function in these parameters. Then the *degree* of the term is the degree of this function and represents its order in respect to the eccentricities and inclinations.

A further classification is used by Poincaré. The order of a term being α, the *rank* of a term is represented by $\alpha - m$, or by the order less the exponent of t. A term of high order is initially small, but if m is large it will grow rapidly in importance, so that ultimately the terms of the lowest rank will have the greatest significance.

The occurrence of long-period terms with small divisors has been noticed. In the higher orders these divisors will be combined and raised to higher powers by the subsequent integrations. Let m' be the sum of the exponents of such divisors in any term. Then the *class* of that term is defined by the number $\alpha - \frac{1}{2}(m + m')$. It will now be clear that the value of these different categories depends on the length of time contemplated. For relatively short

intervals the most important terms are those of low order. In longer intervals the terms of low class rise into prominence. And finally it is the terms of low rank which have the greatest influence in the ultimate destiny of the system.

But here a question naturally arises. How far is the form in which the terms present themselves natural to the problem, and how far are they the artificial product of the particular method by which they are obtained? It is evident that the physical importance of this question is not quite the same in all cases. Thus a mean motion in the position of the node or perihelion may be admitted without any serious direct consequences to the nature of the system. On the other hand, a purely secular term in the mean distance or the eccentricity, taken by itself without compensating circumstances, must ultimately prove fatal to the stability. The general problem suggested is very difficult and the reader is referred to the first volume of Poincaré's *Leçons de Mécanique Céleste* for a thorough discussion.

It must, however, be pointed out that the form of the results may be perfectly legitimate, so far as it goes, and at the same time not in any way inconsistent with the stability of the system, though a decision is beyond the range of the above elementary methods. It is impossible to be satisfied with the solution here described as a final representation, and this feeling is obviously suggested by considering the mixed terms. Since the corresponding oscillations increase in amplitude indefinitely with the time the departure from the original configuration will become so great that the fundamental assumption of small displacements in forming the equations for the variations will be contravened. Then one of two things will happen. Either the mutual forces will tend to restore the original configuration, and there will be stability, or the forces will tend to magnify the disturbance, and there will be instability. But in either case equally the method adopted breaks down and the fundamental question remains unanswered.

How then are the statements to be reconciled, that the method—which is the method on which the existing theories of the major planets are actually based—may be perfectly legitimate, and that, while the form of the terms to which it leads obviously suggests instability, complete stability is nevertheless entirely possible? The simple answer is that it is only necessary to imagine that ν in the argument of any term is itself a function of the disturbing masses. Now the above method involves a development in powers of the masses, and when the parameters which represent the masses are thus forced out of the circular functions they carry the time t explicitly with them, and the appearance of secular and mixed terms is a natural consequence. Yet the development in terms of the masses may be convergent and entirely legitimate. In this way it will be seen that the occurrence of secular and mixed terms is compatible with stability, though a profound discussion is necessary for a positive conclusion on this point.

The case of a planet moving in a resisting medium is quite different. There is then a definite loss of energy and the effect of the secular changes is not doubtful.

168. In the theories of the planets on which the existing tables have been based the coordinates of the planets relative to the Sun have been used and this fact governs the form of the disturbing function, which is distinct for each pair of planets. For practical purposes this choice of coordinates is an obvious one. But for theoretical purposes it is unsuitable, chiefly because, like the common system of elliptic elements, it is ill adapted to the transformations which are an essential feature of the dynamical methods initiated by Hamilton. Another system of coordinates, due to Jacobi, will therefore now be introduced.

Let (ξ_i, η_i, ζ_i) be the coordinates of the mass m_i in a system of n masses $m_1, m_2, \ldots, m_n$, the origin being any fixed point. The masses are taken in any fixed order, represented by the suffixes, which is quite independent of any arrangement which may be visible in the system. Let

$$m_1 + m_2 + \ldots + m_i = \mu_i, \quad m_i = \mu_i - \mu_{i-1}, \quad \mu_0 = 0.$$

Let (X_i, Y_i, Z_i) be the coordinates of the point G_i, which is the centre of mass of the partial system $m_1, m_2, \ldots, m_i$, so that

$$\mu_i X_i = \mu_1 \xi_1 + (\mu_2 - \mu_1)\,\xi_2 + \ldots + (\mu_i - \mu_{i-1})\,\xi_i$$

$$(\mu_i - \mu_{i-1})\,\xi_i = \mu_i X_i - \mu_{i-1} X_{i-1}, \quad \xi_1 = X_1.$$

Let (x_i, y_i, z_i) be the coordinates of m_i relative to G_{i-1}, so that

$$x_i = \xi_i - X_{i-1}, \quad (\mu_i - \mu_{i-1})\,x_i = \mu_i (X_i - X_{i-1}).$$

Thus (x_2, y_2, z_2) are the coordinates of m_2 relative to m_1, or $(\xi_2 - \xi_1, \eta_2 - \eta_1, \zeta_2 - \zeta_1)$; (x_3, y_3, z_3) are the coordinates of m_3 relative to G_2, the centre of mass of m_1, m_2; and so on. There are no coordinates (x_1, y_1, z_1). By the above

$$(\mu_i - \mu_{i-1})^2\,\xi_i^2 = (\mu_i X_i - \mu_{i-1} X_{i-1})^2$$

$$(\mu_i - \mu_{i-1})^2\,x_i^2 = \mu_i^2 (X_i - X_{i-1})^2.$$

Hence on eliminating the product term $X_i X_{i-1}$

$$(\mu_i - \mu_{i-1})\,(\xi_i^2 - \mu_{i-1} x_i^2/\mu_i) = \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2$$

and on addition of all the equations of this type

$$\sum_{i=1}^{n} (\mu_i - \mu_{i-1})\,(\xi_i^2 - \mu_{i-1} x_i^2/\mu_i) = \mu_n X_n^2$$

$$\sum_{i=1}^{n} m_i \xi_i^2 = \sum_{i=2}^{n} m_i \mu_{i-1} x_i^2/\mu_i + \mu_n X_n^2.$$

The relations between the coordinates have been written down for one only. But they are linear and the same for all three coordinates separately.

Therefore they also apply to the velocities. Hence if T is the kinetic energy of the system,

$$\begin{aligned}2T &= \sum_{i=1}^{n} m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2)\\ &= \sum_{i=2}^{n} m_i\mu_{i-1}\mu_i^{-1}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) + \mu_n(\dot{X}_n^2 + \dot{Y}_n^2 + \dot{Z}_n^2).\end{aligned}$$

But (X_n, Y_n, Z_n) are the coordinates of the centre of mass of the system. They are absent from the potential function and are in fact ignorable coordinates. The known integrals for the centre of mass follow immediately and these coordinates can be suppressed. The problem of n bodies is thus reduced to a problem of $n-1$ fictitious bodies and the total order of the differential equations of motion is reduced by 6.

169. The new form of the areal integrals is easily found. For

$$\begin{aligned}(\mu_i - \mu_{i-1})^2(\eta_i\dot{\zeta}_i - \zeta_i\dot{\eta}_i) &= (\mu_i Y_i - \mu_{i-1}Y_{i-1})(\mu_i\dot{Z}_i - \mu_{i-1}\dot{Z}_{i-1})\\ &\quad - (\mu_i Z_i - \mu_{i-1}Z_{i-1})(\mu_i\dot{Y}_i - \mu_{i-1}\dot{Y}_{i-1})\end{aligned}$$

$$(\mu_i - \mu_{i-1})^2(y_i\dot{z}_i - z_i\dot{y}_i) = \mu_i^2(Y_i - Y_{i-1})(\dot{Z}_i - \dot{Z}_{i-1}) - \mu_i^2(Z_i - Z_{i-1})(\dot{Y}_i - \dot{Y}_{i-1})$$

and hence

$$\begin{aligned}(\mu_i - \mu_{i-1})\{(\eta_i\dot{\zeta}_i - \zeta_i\dot{\eta}_i) - \mu_{i-1}\mu_i^{-1}(y_i\dot{z}_i - z_i\dot{y}_i)\}&\\ = \mu_i(Y_i\dot{Z}_i - Z_i\dot{Y}_i) - \mu_{i-1}(Y_{i-1}\dot{Z}_{i-1} - Z_{i-1}\dot{Y}_{i-1}).&\end{aligned}$$

The sum of all equations of this type gives

$$\sum_{i=1}^{n} m_i\{(\eta_i\dot{\zeta}_i - \zeta_i\dot{\eta}_i) - \mu_{i-1}\mu_i^{-1}(y_i\dot{z}_i - z_i\dot{y}_i)\} = \mu_n(Y_n\dot{Z}_n - Z_n\dot{Y}_n).$$

But it is possible to write $X_n = Y_n = Z_n = 0$; that is equivalent to taking the centre of mass of the system as the origin of the coordinates (ξ_i, η_i, ζ_i). Thus the areal integrals now take the form

$$\sum_{i=2}^{n} m_i\mu_{i-1}\mu_i^{-1}(y_i\dot{z}_i - z_i\dot{y}_i) = c_1$$

$$\sum_{i=2}^{n} m_i\mu_{i-1}\mu_i^{-1}(z_i\dot{x}_i - x_i\dot{z}_i) = c_2$$

$$\sum_{i=2}^{n} m_i\mu_{i-1}\mu_i^{-1}(x_i\dot{y}_i - y_i\dot{x}_i) = c_3$$

where (c_1, c_2, c_3) are the angular momenta of the system about fixed axes through the centre of mass. The direction of the axes has remained the same throughout.

Let (c_1, c_2, c_3) be considered as the components of a constant vector C, $m_i\mu_{i-1}\mu_i^{-1}(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ as the components of a vector M_i, and (x_i, y_i, z_i) as the

components of a vector r_i. Then in quaternion notation the above three integrals may be represented by the single equation

$$\sum_{i=2}^{n} V(r_i M_i) = C.$$

Hence in the problem of three bodies

$$V(r_2 M_2) + V(r_3 M_3) = C.$$

These three vectors are therefore coplanar. But $V(r_2 M_2)$ is normal to the plane of r_2, M_2, that is, to the instantaneous orbit of the fictitious planet 2. Similarly $V(r_3 M_3)$ is normal to the instantaneous orbit of the fictitious planet 3, and clearly C is normal to the invariable plane. Hence the nodes of the instantaneous orbits of the two fictitious planets on the invariable plane coincide.

This important property explains the so-called *elimination of the nodes*, which in an explicit form is due to Jacobi. In the more common system of astronomical coordinates it disappears from view. The reader who is unacquainted with the elements of quaternions will have no difficulty in finding an alternative form of proof, as in § 22.

170. The body denoted by 1 will now be identified with the Sun, and i or j will have the values $2, \ldots, n$. The potential energy of the system, when the units are chosen so that the constant of gravitation is unity, is

$$U = -\Sigma \frac{m_1 m_i}{\Delta_{1,i}} - \Sigma \frac{m_i, m_j}{\Delta_{i,j}}$$

where

$$\Delta_{i,j}^2 = (\xi_j - \xi_i)^2 + (\eta_j - \eta_i)^2 + (\zeta_j - \zeta_i)^2.$$

Also the kinetic energy, when the coordinates (X_n, Y_n, Z_n) are ignored, is T, where

$$2T = \sum_{i=2}^{n} m_i \mu_{i-1} \mu_i^{-1} (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2).$$

Let

$$x_i' = \frac{\partial T}{\partial \dot{x}_i} = m_i \mu_{i-1} \mu_i^{-1} \dot{x}_i, \ldots, \quad H = T + U.$$

Then the equations of motion of the system may be written (§ 124)

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i'}, \quad \frac{dx_i'}{dt} = -\frac{\partial H}{\partial x_i}, \quad (x, y, z).$$

Now

$$(\mu_i - \mu_{i-1}) \xi_i = \mu_i X_i - \mu_{i-1} X_{i-1} = \mu_i (\xi_{i+1} - x_{i+1}) - \mu_{i-1} (\xi_i - x_i)$$

and therefore

$$\xi_{i+1} - \xi_i = x_{i+1} - \mu_{i-1} x_i / \mu_i.$$

Hence by the addition of such equations

$$\xi_{i+1} - \xi_1 = x_{i+1} + m_i x_i / \mu_i + \ldots + m_2 x_2 / \mu_2, \quad \xi_2 - \xi_1 = x_2$$

which expresses the relative coordinates $\xi_i - \xi_1, \ldots$ in terms of the coordinates $x_i, \ldots$, and shows that the latter differ from the former only by quantities of the first order in the small masses. In particular, for the body 2, which may be identified with any one of the planets, there is no difference.

Let U be reduced to its terms U_1 of the lowest order in the small masses, which is the first. Then

$$U_1 = -m_1 \Sigma\, m_i/r_i, \quad r_i^2 = x_i^2 + y_i^2 + z_i^2$$

for r_i differs from $\Delta_{1,i}$ by a quantity which involves the masses. The equations of motion reduce to

$$\frac{dx_i}{dt} = \frac{\partial H_1}{\partial x_i'}, \quad \frac{dx_i'}{dt} = -\frac{\partial H_1}{\partial x_i}, \quad H_1 = T + U_1$$

or in more explicit form

$$\mu_{i-1}\mu_i^{-1}\, \ddot{x}_i = -m_1 x_i/r_i^3, \quad (x, y, z).$$

These are the equations of undisturbed elliptic motion, and in particular

$$\ddot{x}_2 = -(m_1 + m_2)\, x_2/r_2^3, \quad (x, y, z)$$

which agree naturally with the usual equations of a planet relative to the Sun in undisturbed motion, and give a mean distance a_2 with the usual meaning. For the other bodies the equations are of the same form and have precisely similar solutions, but the elements a_i will differ from the ordinary elements slightly because (x_i, y_i, z_i) are not coordinates relative to the Sun unless $i = 2$. This is not material to the purpose in view because the body 2 represents any planet and any proposition which is proved for it must be true generally.

171. These equations for the undisturbed motion can now be solved in terms of canonical constants. When the latter are treated as variables, they satisfy canonical equations formed with $R = U_1 - U$. As in § 143 this value of R may be modified by adding $\Sigma\, m\mu^2/2L'^2$, where $m = m_i\mu_{i-1}/\mu_i$ and $\mu = m_1\mu_i/\mu_{i-1}$ in view of the explicit form of the undisturbed equations. Then any of the different sets of variables explained in that section can be used, and the last set, now denoted by $(L', \xi_1', \xi_2';\ \lambda, \eta_1', \eta_2')$, will be chosen. The equations for the perturbations can now be written

$$\frac{m_i\mu_{i-1}}{\mu_i}\frac{dL_i'}{dt} = \frac{\partial V}{\partial \lambda_i}, \quad \frac{m_i\mu_{i-1}}{\mu_i}\frac{d\lambda_i}{dt} = -\frac{\partial V}{\partial L_i'}$$

$$\frac{m_i\mu_{i-1}}{\mu_i}\frac{d\xi_i'}{dt} = \frac{\partial V}{\partial \eta_i'}, \quad \frac{m_i\mu_{i-1}}{\mu_i}\frac{d\eta_i'}{dt} = -\frac{\partial V}{\partial \xi_i'}$$

where

$$V = -U + U_1 + m_1^2\, \Sigma\, m_i\mu_i/2\mu_{i-1}L_i'^2.$$

There are $n-1$ pairs of equations in (L_i', λ_i) and $2(n-1)$ pairs in (ξ_i', η_i'), but there is no need here to distinguish between the eccentric and oblique

variables. From this point the former use of (ξ_i, η_i, ζ_i) as the rectangular coordinates of m_i disappears.

A little explanation may be necessary to account for the appearance of the mass factors of the momenta x_i' in the equations. In § 135 giving the Hamilton-Jacobi solution for undisturbed elliptic motion the single factor m, representing the mass of the moving body, was removed consistently from U, T and H. Similarly in § 139 $U-R$ was written in the place of U, R being the disturbing function in its common form, whereas the true increment in the potential energy is $-mR$. But here it is not possible to divide the more general function $U-U_1$ as a whole by any particular mass, though it is possible to do so as regards the set of equations corresponding to a particular value of i. Hence it was necessary to restore the mass factors in the manner shown. But now they can be removed by the change of variables,

$$L_i = \frac{m_i\mu_{i-1}}{\mu_i} L_i', \quad \xi_i = \left(\frac{m_i\mu_{i-1}}{\mu_i}\right)^{\frac{1}{2}} \xi_i', \quad \eta_i = \left(\frac{m_i\mu_{i-1}}{\mu_i}\right)^{\frac{1}{2}} \eta_i'$$

and the equations then become

$$\frac{dL_i}{dt} = \frac{\partial V}{\partial \lambda_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial V}{\partial L_i}$$

$$\frac{d\xi_i}{dt} = \frac{\partial V}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial V}{\partial \xi_i}$$

where

$$V = -U + U_1 + m_1^2 \Sigma\, m_i^3\mu_{i-1}/2\mu_i L_i^2.$$

The terms added to $U_1 - U$ depend on the L_i only, and affect one type of equation, namely

$$\frac{d\lambda_i}{dt} = \frac{\partial}{\partial L_i}(U - U_1) + \frac{m_1^2 m_i^3 \mu_{i-1}}{\mu_i L_i^3} = \frac{\partial}{\partial L_i}(U - U_1) + n_i$$

so that $\lambda_i = n_i t + h$ and n_i is the mean motion in the preliminary solution. The first-order perturbations of λ_i will require the first-order perturbation of L_i to be included in the term from which n_i originates.

172. It is not at present very necessary to consider in detail the form of expansion of $U-U_1$. It can in the first place be expanded in powers and products of the small masses m_i and of the coordinates (x_i, y_i, z_i). The latter can be expanded in powers of L_i, ξ_i, η_i with purely periodic functions of λ_i. Hence $U-U_1$ can be expanded in the same form, and arranged in orders of the masses, beginning with the second since the first has been removed by U_1. Thus if the fourth order in V be neglected, $V = V_2 + V_3$, where V_2 is of the second order and V_3 of the third, and V_2 contains at most two, V_3 at most three, mean longitudes λ_i in its arguments, the coefficients of the periodic terms being rational and integral functions of L_i, ξ_i, η_i.

The perturbations of the first order can now be obtained in the usual way by neglecting V_3 and substituting initial values of L_i, ξ_i, η_i in V_2, including $n_i t + \lambda_i^0$ for λ_i. This process gives

$$L_i = L_i^0 + \delta_1 L_i^0, \quad \lambda_i = n_i t + \lambda_i^0 + \delta_1 \lambda_i^0, \quad \xi_i = \xi_i^0 + \delta_1 \xi_i^0, \quad \eta_i = \eta_i^0 + \delta_1 \eta_i^0$$

where $L_i^0, \ldots$ are constants and $\delta_1 L_i^0, \ldots$ are the perturbations of the first order. Owing to the form of V_2, $\partial V_2/\partial\lambda_i$ is purely periodic and free from any term independent of λ_i. Hence $\delta_1 L_i^0$ is also periodic and free from a secular term. But the other elements will contain a term multiplied by t, arising from the terms independent of λ_i in the partial derivatives of V_2, together with periodic terms. To the second order let

$$L_i = L_i^0 + \delta_1 L_i^0 + \delta_2 L_i^0.$$

In V_3, which must now be retained, it suffices to substitute the constant values $L_i^0, \ldots$ for $L_i, \ldots$, and $n_i t + \lambda_i^0$ for λ_i; but in V_2 it is necessary to substitute $L_i^0 + \delta_1 L_i^0, \ldots$ for $L_i, \ldots$, though only the first powers of these perturbations are required. Hence the equation

$$\frac{d}{dt}(L_i^0 + \delta_1 L_i^0 + \delta_2 L_i^0) = \frac{\partial}{\partial\lambda_i}(V_2 + V_3)$$

gives, when account is taken of the solution for the first order,

$$\frac{d}{dt}(\delta_2 L_i^0) = \sum_j \left(\frac{\partial^2 V_2}{\partial\lambda_i \partial L_j^0}\delta_1 L_j^0 + \frac{\partial^2 V_2}{\partial\lambda_i\partial\lambda_j}\delta_1\lambda_j^0 + \ldots\right) + \frac{\partial V_3}{\partial\lambda_i}.$$

By the same argument as applied to V_2 in the first approximation the last term gives rise to periodic terms only. Hence a search for secular terms can be confined in the first place to the expression

$$\sum_j \left[\frac{\partial^2 V_2}{\partial\lambda_i\partial L_j^0}\int\frac{\partial V_2}{\partial\lambda_j}dt - \frac{\partial^2 V_2}{\partial\lambda_i\partial\lambda_j}\int\frac{\partial V_2}{\partial L_j^0}dt + \frac{\partial^2 V_2}{\partial\lambda_i\partial\xi_j^0}\int\frac{\partial V_2}{\partial\eta_j^0}dt - \frac{\partial^2 V_2}{\partial\lambda_i\partial\eta_j^0}\int\frac{\partial V_2}{\partial\xi_j^0}dt\right].$$

Here the multipliers of the integrals are all purely periodic, owing to differentiation with respect to λ_i. The integrals themselves contain secular terms in t. Hence on integration the products will give rise to periodic and mixed terms, but not to purely secular terms on this account. The latter must arise, if at all, from a constant term in the products. The only way in which this could happen would be connected with terms in the development of V_2 of the form

$$V_2 = B\sin(k_i\lambda_i + k_j\lambda_j) + C\cos(k_i\lambda_i + k_j\lambda_j) = B\sin\psi + C\cos\psi.$$

But for these

$$\frac{\partial^2 V_2}{\partial\lambda_i\partial L_j^0}\int\frac{\partial V_2}{\partial\lambda_j}dt - \frac{\partial^2 V_2}{\partial\lambda_i\partial\lambda_j}\int\frac{\partial V_2}{\partial L_j^0}dt$$

$$= k_i\left(\frac{\partial B}{\partial L_j^0}\cos\psi - \frac{\partial C}{\partial L_j^0}\sin\psi\right)\cdot\frac{k_j}{k_i n_i + k_j n_j}(B\sin\psi + C\cos\psi)$$

$$+ k_i k_j(B\sin\psi + C\cos\psi)\cdot\frac{1}{k_i n_i + k_j n_j}\left(-\frac{\partial B}{\partial L_j^0}\cos\psi + \frac{\partial C}{\partial L_j^0}\sin\psi\right)$$

$$= 0.$$

In a similar way those terms which might produce constant terms neutralize one another between the other pairs of products and therefore no purely secular part of $\delta_2 L_i^0$ can arise in this way.

But the above expression is not complete, because $\delta_1 \lambda_j^0$ depends on $\delta_1 L_j^0$ as well as on V_2. For, by the last equation of § 171,

$$\frac{d\delta_1\lambda_j^0}{dt} = -\frac{\partial V_2}{\partial L_j^0} - \frac{3m_1^2 m_j^3 \mu_{j-1}}{\mu_j (L_j^0)^4} \delta_1 L_j^0$$

so that there is an additional part of $\delta_2 L_i^0$ not yet considered. It is given by

$$\frac{d}{dt}(\delta_2 L_i^0) = A \sum_j \frac{\partial^2 V_2}{\partial\lambda_i \partial\lambda_j} \int \delta_1 L_j^0 \, dt = A \sum_j \frac{\partial^2 V_2}{\partial\lambda_i \partial\lambda_j} \int dt \int \frac{\partial V_2}{\partial\lambda_j} dt$$

where A is a constant. But terms in V_2 of the above type, taken in the form $D \sin(\psi + h)$, lead to

$$\begin{aligned} \frac{d}{dt}(\delta_2 L_i^0) &= A \,.\, k_i k_j D \sin(\psi + h) \,.\, \frac{k_j}{(k_i n_i + k_j n_j)^2} D \cos(\psi + h) \\ &= \frac{A k_i k_j^2}{2(k_i n_i + k_j n_j)^2} D^2 \sin 2(\psi + h). \end{aligned}$$

Therefore this part of $\delta_2 L_i^0$ is purely periodic.

Hence there are no purely secular terms in $\delta_2 L_i^0$, a proposition which Poincaré has proved in the more general form: there are no purely secular perturbations of L_i in any order of rank lower than 2.

This applies in particular to L_2. But $a_2 = ML_2^2$, where M is a constant mass factor. Hence

$$a_2 + \delta_1 a_2 + \delta_2 a_2 = M(L_2 + \delta_1 L_2 + \delta_2 L_2)^2$$

$$\delta_1 a_2 = 2ML_2 \delta_1 L_2, \quad \delta_2 a_2 = M\{(\delta_1 L_2)^2 + 2L_2(\delta_2 L_2)\}$$

the affix ^{0} being now omitted. But $\delta_1 L_2$ is purely periodic, and $\delta_2 L_2$ has no purely secular term. Hence to the second order in the masses there is no secular inequality in the mean distance, for it has been remarked that a_2 represents the mean distance of any of the planets. This is Poisson's theorem, an extension of Laplace's corresponding theorem for the first order, and it is the most important elementary result bearing on the stability of the solar system.

173. On the other hand there are evidently mixed terms of order 2 and rank 1 in L_i. Hence the existence of purely secular terms of order 3 and rank 2 in a_2 can be anticipated. For even without pushing the approximation further and examining $\delta_3 L_2$ it is obvious that $2M\delta_1 L_2 \,.\, \delta_2 L_2$ constitutes a part of $\delta_3 a_2$. Therefore the combination of a term $A \cos mt$ in $\delta_1 L_2$ with a term $Bt \cos mt$ in $\delta_2 L_2$ will give a term $MABt$ in $\delta_3 a_2$. Such terms were first shown to exist by Spiru-Haretu in 1876.

On one condition true secular inequalities of the first order occur in the mean distances. Since

$$U - U_1 = \Sigma\, A \cos(k_i\lambda_i + k_j\lambda_j + h)$$

to its lowest order,

$$\partial V/\partial\lambda_i = \Sigma\, Ak_i \sin(k_i\lambda_i + k_j\lambda_j + h).$$

For perturbations of the first order the coefficients are constants and $\lambda_i - n_i t$, $\lambda_j - n_j t$ are also constant. Hence

$$dL_i/dt = \Sigma\, Ak_i \sin(mt + h').$$

A constant term results, producing a secular inequality, if $m = k_i n_i + k_j n_j = 0$, which is possible only if n_i, n_j are commensurable. This possibility was considered in the previous form of discussion and excluded. But it is in effect ruled out by its own consequences. For if a body were artificially or fortuitously projected in such a way as to have a mean motion commensurable (e.g. $\frac{1}{2}$, $\frac{2}{3}$, ...) with the mean motion of a disturbing body, its mean distance would be subject to a secular disturbance from the beginning, and therefore the commensurability of its motion would be definitely destroyed. Hence if the minor planets be arranged in order of distance from the Sun, it is to be expected that gaps will be found in the frequency at distances corresponding to mean motions commensurable with that of Jupiter, and it is so. And similarly divisions in the rings of Saturn can be attributed to the secular perturbations of the constituent meteoric bodies, produced by the commensurable motions of any satellite which may be effective. This also has been verified for the action of Mimas by Lowell and Slipher.

Nevertheless among the many minor planets a few are naturally found whose motions are nearly commensurable with Jupiter's mean motion. For these the long-period terms with small divisors are highly important, and the terms of low *class* play a far larger part than in the theories of the major planets. The special difficulties thus presented require special methods of treatment, and such have been suggested by Hansen, Gyldén and others. Poincaré has used an application of the principle of Delaunay's method. The proper treatment of this class of minor planets presents perhaps the most interesting problems to be found in dynamical Astronomy at the present time.

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CHAPTER XVI

SECULAR PERTURBATIONS

174. In the preceding chapter it has been shown that the mean distances in the planetary system are free from purely secular inequalities when developed to the second order in the masses. The general nature of the secular perturbations in the other elements will now be examined. It may be convenient to modify slightly the equations obtained in §§ 170, 171. By reducing U to its terms of the lowest order the equations of motion there took the explicit form

$$\mu_{i-1}\mu_i^{-1}\ddot{x}_i = -m_1 x_i/r_i^3, \qquad (x, y, z)$$

which are satisfied by the osculating motion of a planet, according to its ordinary definition, when $i=2$, but not otherwise. But if U_1' be substituted for U_1, where

$$U_1' = -\Sigma\,(m_1+m_i)\,m_i\mu_{i-1}/\mu_i r_i$$

a form which will be found to differ from U_1 by terms of the third order only, the explicit equations of motion become

$$\ddot{x}_i = -(m_1+m_i)\,x_i/r_i^3, \qquad (x, y, z)$$

which are the ordinary equations in the undisturbed problem of two bodies, and are satisfied by the osculating elements taken in their usual sense. The mass factors of the momenta are as before $m_i\mu_{i-1}/\mu_i$, but the constants of attraction are $\mu = m_1 + m_i$. Hence the equations for the variations will now be based on

$$V' = -U + U_1' + \Sigma\,(m_1+m_i)^2\,m_i\mu_{i-1}/2\mu_i L_i'^2$$

$$= -U + U_1' + \Sigma\,(m_1+m_i)^2\,m_i^3\mu^3_{i-1}/2\mu_i^3 L_i^2.$$

The relation between L_i and L_i' is the same as before, but the meaning of both is changed (except when $i=2$) in such a way that L_i bears generally the same form of relation to a_i, the osculating mean distance in its ordinary sense, as L_2 to a_2.

Thus the transformations of § 143 give, with those of § 171,

$$L_i' = (m_1 + m_i)^{\frac{1}{2}} a_i^{\frac{1}{2}}, \quad G_i = L_i' \cos \phi_i, \quad H_i = G_i \cos i$$
$$l_i = \epsilon_i - \varpi_i + n_i t, \quad g_i = \varpi_i - \Omega_i, \quad h_i = \Omega_i$$
$$\rho_{i,1} = 2L_i' \sin^2 \tfrac{1}{2}\phi_i, \quad \rho_{i,2} = 2L_i' \cos \phi_i \sin^2 \tfrac{1}{2} i_i$$
$$\lambda_i = \epsilon_i + n_i t, \quad \omega_{i,1} = -\varpi_i, \quad \omega_{i,2} = -\Omega_i$$
$$L_i = m_i (m_1 + m_i)^{\frac{1}{2}} \mu_{i-1} \mu_i^{-1} a_i^{\frac{1}{2}}$$
$$\xi_{i,1} = 2L_i^{\frac{1}{2}} \sin \tfrac{1}{2}\phi_i \cos \varpi_i, \quad \eta_{i,1} = -2L_i^{\frac{1}{2}} \sin \tfrac{1}{2}\phi_i \sin \varpi_i$$
$$\xi_{i,2} = 2L_i^{\frac{1}{2}} \cos^{\frac{1}{2}} \phi_i \sin \tfrac{1}{2} i_i \cos \Omega_i, \quad \eta_{i,2} = -2L_i^{\frac{1}{2}} \cos^{\frac{1}{2}} \phi_i \sin \tfrac{1}{2} i_i \sin \Omega_i.$$

Here $\sin \phi_i = e_i$ and no confusion is possible between the inclination i and the subscript i, which is merely a distinguishing mark for the several planets.

175. It is obvious that $U - U_1'$ can be expanded in powers of $x_i - a_i$, $y_i - b_i$, $z_i - c_i$ where (a_i, b_i, c_i) are what (x_i, y_i, z_i) become when $\xi_i = \eta_i = 0$. Now (§ 65) the heliocentric coordinates are generally

$$x = r \cos \Omega \cos (w + \varpi - \Omega) - r \cos i \sin \Omega \sin (w + \varpi - \Omega)$$
$$= r \cos^2 \tfrac{1}{2} i \cos (w + \varpi) + r \sin^2 \tfrac{1}{2} i \cos (w + \varpi - 2\Omega)$$
$$y = r \sin \Omega \cos (w + \varpi - \Omega) + r \cos i \cos \Omega \sin (w + \varpi - \Omega)$$
$$= r \cos^2 \tfrac{1}{2} i \sin (w + \varpi) - r \sin^2 \tfrac{1}{2} i \sin (w + \varpi - 2\Omega)$$
$$z = r \sin i \sin (w + \varpi - \Omega)$$

w being the true anomaly. Let

$$X = r \cos (w - M), \quad Y = r \sin (w - M), \quad M = \lambda - \varpi$$

M being the mean anomaly. Then

$$x = X \{\cos^2 \tfrac{1}{2} i \cos \lambda + \sin^2 \tfrac{1}{2} i \cos (\lambda - 2\Omega)\}$$
$$- Y \{\cos^2 \tfrac{1}{2} i \sin \lambda + \sin^2 \tfrac{1}{2} i \sin (\lambda - 2\Omega)\}$$
$$y = X \{\cos^2 \tfrac{1}{2} i \sin \lambda - \sin^2 \tfrac{1}{2} i \sin (\lambda - 2\Omega)\}$$
$$+ Y \{\cos^2 \tfrac{1}{2} i \cos \lambda - \sin^2 \tfrac{1}{2} i \cos (\lambda - 2\Omega)\}$$
$$z = X \sin i \sin (\lambda - \Omega) + Y \sin i \cos (\lambda - \Omega).$$

The coefficients of X and Y here involve, besides periodic functions of λ, the quantities

$$\cos^2 \tfrac{1}{2} i, \quad \sin^2 \tfrac{1}{2} i \cos 2\Omega, \quad \sin^2 \tfrac{1}{2} i \sin 2\Omega, \quad \sin i \cos \Omega, \quad \sin i \sin \Omega$$

and since

$$\xi_1^2 + \eta_1^2 = 4L \sin^2 \tfrac{1}{2}\phi, \quad \xi_2^2 + \eta_2^2 = 4L \cos \phi \sin^2 \tfrac{1}{2} i$$
$$\tan \varpi = -\eta_1/\xi_1, \quad \tan \Omega = -\eta_2/\xi_2$$

it is easily verified that the five quantities can all be expanded in powers of $\xi_1, \eta_1, \xi_2, \eta_2$. Also

$$r \cos w = a (\cos E - e), \quad r \sin w = a \cos \phi \sin E$$

E being the eccentric anomaly, and therefore

$$X/a = -e\cos M + \cos^2\tfrac{1}{2}\phi\cos(E-M) + \tfrac{1}{4}\sec^2\tfrac{1}{2}\phi\,\{e^2\cos 2M\cos(E-M) - e^2\sin 2M\sin(E-M)\}$$

$$Y/a = e\sin M + \cos^2\tfrac{1}{2}\phi\sin(E-M) - \tfrac{1}{4}\sec^2\tfrac{1}{2}\phi\,\{e^2\cos 2M\sin(E-M) + e^2\sin 2M\cos(E-M)\}$$

which are forms easily verified. Since $\cos^2\frac{1}{2}\phi$, $\sec^2\frac{1}{2}\phi$ can be expanded in terms of $e^2 = \sin^2\phi$, these forms show that X, Y can be expanded in powers of $e\sin M$, $e\cos M$ if this is true of $\sin(E-M)$, $\cos(E-M)$. But Kepler's equation may be written

$$\theta - x\cos\theta - y\cos\theta = 0, \quad \theta = E - M, \quad x = e\sin M, \quad y = e\cos M$$

and θ can be expanded in powers of x, y. Hence $\sin(E-M)$, $\cos(E-M)$ can be expanded in powers of $e\sin M$, $e\cos M$, and therefore also X and Y. But this shows that X, Y can be expanded in powers of $e\sin\varpi$, $e\cos\varpi$ with coefficients involving periodic functions of λ, since $M = \lambda - \varpi$. And $e\sin\varpi$, $e\cos\varpi$ can be expanded in powers of ξ_1, η_1, as can easily be seen, with coefficients involving L. Hence (x, y, z) can be developed in powers of ξ_1, η_1, ξ_2, η_2 with coefficients involving L and periodic functions of λ. Therefore finally $U - U_1'$ can be expanded in powers of $\xi_{i,1}$, $\eta_{i,1}$, $\xi_{i,2}$, $\eta_{i,2}$ with coefficients involving L_i and periodic functions of λ_i, and the supplementary part of V' involves L_i only.

It is assumed that the inclinations of the orbits are very small. Now there are two ways of regarding retrograde motion in an orbit whose plane differs little from the orbits of planets moving in the opposite sense. It is possible to take the mean motion n_i as positive. Then the inclination is near π and is not small. Or it is possible to take the inclination as small, and to regard n_i as negative. Then since $n_i L_i^3$ is a positive mass function, L_i is negative and therefore ξ_i, η_i are imaginary. All the orbits will therefore be supposed to be described in the same (direct) sense, which is true of the planetary system but not always of the satellite systems.

This remark has an obvious bearing on theories of cosmogony. For if high inclinations and in particular retrograde motions were unstable, such forms of motion would not be permanently maintained. Now the nebular hypothesis of Laplace is very largely based on the observed fact that the planetary motions are nearly coplanar. If, however, such a type of motion is alone stable, the observed fact loses its significance in this connexion and no deduction of the kind is to be drawn from it. The question of stability in general, beyond the range of inclinations to be found in the actual planetary system, is therefore important, though beyond the range of this work.

176. When the secular part

$$[-U+U_1'] = \Sigma A \xi_{i,1}^{p_1} \eta_{i,1}^{q_1} \xi_{i,2}^{p_2} \eta_{i,2}^{q_2} \dots$$

which is free from λ_i is considered, certain properties of the development are easily seen. For this being independent of the direction of the chosen axes, the substitutions

	$\xi_{i,1}$,	$\eta_{i,1}$,	$\xi_{i,2}$,	$\eta_{i,2}$
(*a*)	$-\xi_{i,1}$,	$-\eta_{i,1}$,	$-\xi_{i,2}$,	$-\eta_{i,2}$
(*b*)	$\eta_{i,1}$,	$-\xi_{i,1}$,	$\eta_{i,2}$,	$-\xi_{i,2}$
(*c*)	$\xi_{i,1}$,	$\eta_{i,1}$,	$-\xi_{i,2}$,	$-\eta_{i,2}$
(*d*)	$\xi_{i,1}$,	$-\eta_{i,1}$,	$\xi_{i,2}$,	$-\eta_{i,2}$

are all possible without affecting the result. Thus (*a*) follows when Ω_i, ϖ_i are altered by π, or when the axes of xy are rotated through π in their own plane. Similarly (*b*) follows when this rotation is made through $\frac{1}{2}\pi$. Again (*c*) is produced when Ω_i (but not ϖ_i) is altered by π, and this is equivalent to reversing the axis of z. Finally (*d*) is obtained by changing the signs of all the angles λ_i, Ω_i, ϖ_i, which is equivalent to reversing the axis of y. The change in λ_i is of no further importance here since λ_i is absent from the terms now considered.

Certain properties of the exponents in the expansion are now obvious. For $\Sigma(p_1+q_1+p_2+q_2)$ must be an even number to satisfy (*a*), and $\Sigma(p_2+q_2)$ to satisfy (*c*). Hence $\Sigma(p_1+q_1)$ is also an even number. Similarly (*d*) requires $\Sigma(q_1+q_2)$ to be even, and therefore $\Sigma(p_1+p_2)$ must be even. Hence in the second degree there can be no terms of the form $\xi\eta$ or. $\xi_1\xi_2$, $\eta_1\eta_2$. But if terms of the fourth degree be neglected, only terms of the second degree involving ξ, η remain. These terms can therefore be written down in the form

$$[-U+U_1'] = \Sigma \tfrac{1}{2} A_{i,j}(\xi_{i,1}\xi_{j,1}+\eta_{i,1}\eta_{j,1}) + \Sigma\tfrac{1}{2}B_{i,j}(\xi_{i,2}\xi_{j,2}+\eta_{i,2}\eta_{j,2})$$

where the coefficients of $\xi_i\xi_j$, $\eta_i\eta_j$ are taken to be the same, both for the eccentric and the oblique variables, in accordance with the substitution (*b*), and terms $\xi_i\xi_j$, $\eta_i\eta_j$ are reckoned twice when i, j are different, but $A_{i,j}=A_{j,i}$, $B_{i,j}=B_{j,i}$.

177. It will be of interest to obtain the explicit values of $A_{i,j}$, $B_{i,j}$ for the lowest order in the masses. The principal part of the disturbing function is $\Sigma m_i m_j \Delta_{i,j}^{-1}$ and it has been seen in § 159 that the complementary part contains periodic terms only. The distances $\Delta_{i,j}$ involve coordinates (x_i, y_i, z_i) which themselves contain the masses. But to the lowest order these coordinates are identical with the relative coordinates commonly in use, and the methods of Chap. XIV can therefore be employed. Two planets, 1, 2, will be first considered. Then in the notation of § 152, when the orbits are circular,

$$a_2\Delta^{-1} = b^{0,0} = \tfrac{1}{2}b_{\frac{1}{2}}^{0} - \tfrac{1}{2}\alpha\nu b_{\frac{3}{2}}^{1} + \dots$$

with the exclusion of all periodic terms. The triangle formed by the two orbits and the ecliptic gives

$$\cos J = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_1 - \Omega_2)$$

or to the second order in i_1, i_2,

$$\nu = \sin^2 \tfrac{1}{2} J = \tfrac{1}{4}\{i_1^2 + i_2^2 - 2i_1 i_2 \cos(\Omega_1 - \Omega_2)\}.$$

Since ν is of the second order the Laplace's coefficient $b_{\frac{3}{2}}^1$ is derived immediately from the circular motion. But $b_{\frac{1}{2}}^0$ must be modified to include the eccentricities, the orbits being now treated as coplanar. Let

$$\Delta_0^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos\theta, \quad \theta = \varpi_1 - \varpi_2 + M_1 - M_2.$$

Then in the notation of § 157,

$$\Delta^{-1} = \left(\frac{r_1}{a_1}\right)^{D_1} \left(\frac{r_2}{a_2}\right)^{D_2} \exp.\{(w_1 - M_1) D_3 + (w_2 - M_2) D_4\} \Delta_0^{-1}$$

and, by (22) of § 40 and (30) of § 41,

$$r/a = 1 + \tfrac{1}{2}e^2 - e\cos M - \tfrac{1}{2}e^2 \cos 2M + \dots$$

$$w - M = 2e \sin M + \tfrac{5}{4}e^2 \sin 2M + \dots.$$

Hence

$$(a^{-1}r)^D = 1 - e\cos M \,.\, D + \tfrac{1}{2}e^2(1 - \cos 2M)\, D + \tfrac{1}{4}e^2(1 + \cos 2M)\,.\, D(D-1)$$

$$\exp.(w - M)\, D = 1 + 2e \sin M \,.\, D + \tfrac{5}{4}e^2 \sin 2M \,.\, D + e^2(1 - \cos 2M)\,.\, D^2.$$

All operating terms which do not combine M_1, M_2 in the form $M_1 - M_2$ will clearly produce periodic terms only. And terms already of the second degree are combined with no others. Therefore, when ineffective terms are omitted, since $D_1 + D_2 = -1$,

$$\begin{aligned}
\Delta^{-1} &= (1 - e_1 \cos M_1 \,.\, D_1 - \tfrac{1}{4}e_1^2 \,.\, D_1 D_2)(1 - e_2 \cos M_2 \,.\, D_2 - \tfrac{1}{4}e_2^2 \,.\, D_1 D_2) \\
&\qquad (1 + 2e_1 \sin M_1 \,.\, D_3 + e_1^2 \,.\, D_3^2)(1 + 2e_2 \sin M_2 \,.\, D_4 + e_2^2 \,.\, D_4^2)\, \Delta_0^{-1} \\
&= \{1 + \tfrac{1}{2}e_1 e_2 \cos(M_1 - M_2)\,.\, D_1 D_2 + 2e_1 e_2 \cos(M_1 - M_2)\,.\, D_3 D_4 \\
&\qquad - e_1 e_2 \sin(M_2 - M_1)\,.\, D_1 D_4 - e_1 e_2 \sin(M_1 - M_2)\,.\, D_2 D_3 \\
&\qquad - \tfrac{1}{4}(e_1^2 + e_2^2)\, D_1 D_2 + e_1^2 \,.\, D_3^2 + e_2^2 \,.\, D_4^2\}\, \Delta_0^{-1}
\end{aligned}$$

where again terms involving M_1, M_2 or $M_1 + M_2$ are omitted. Now $D_3 = -D_4 = \partial/\partial\theta$ and, since $\alpha = a_1/a_2$,

$$\begin{aligned}
D_1 D_2 \Delta_0^{-1} &= a_1 a_2 \cos\theta \,.\, \Delta_0^{-3} + 3(a_1^2 - a_1 a_2 \cos\theta)(a_2^2 - a_1 a_2 \cos\theta)\, \Delta_0^{-5} \\
&= a_2^{-1}\{\alpha \cos\theta \,.\, a_2^3 \Delta_0^{-3} + 3\alpha\, [\tfrac{3}{2}\alpha - (1 + \alpha^2)\cos\theta + \tfrac{1}{2}\alpha \cos 2\theta]\, a_2^5 \Delta_0^{-5}\} \\
D_3^2 \Delta_0^{-1} = D_4^2 \Delta_0^{-1} &= -D_3 D_4 \Delta_0^{-1} = -a_1 a_2 \cos\theta \,.\, \Delta_0^{-3} + 3a_1^2 a_2^2 \sin^2\theta \,.\, \Delta_0^{-5} \\
&= a_2^{-1}\{-\alpha\cos\theta \,.\, a_2^3 \Delta_0^{-3} + \tfrac{3}{2}\alpha^2(1 - \cos 2\theta)\,.\, a_2^5 \Delta_0^{-5}\} \\
D_1 D_4 \Delta_0^{-1} &= a_1 a_2 \sin\theta \,.\, \Delta_0^{-3} - 3a_1 a_2 \sin\theta\, (a_1^2 - a_1 a_2 \cos\theta)\, \Delta_0^{-5} \\
&= a_2^{-1}\{\alpha \sin\theta \,.\, a_2^3 \Delta_0^{-3} - 3\alpha^2(\alpha \sin\theta - \tfrac{1}{2}\sin 2\theta)\, a_2^5 \Delta_0^{-5}\} \\
D_2 D_3 \Delta_0^{-1} &= -a_1 a_2 \sin\theta \,.\, \Delta_0^{-3} + 3a_1 a_2 \sin\theta\, (a_2^2 - a_1 a_2 \cos\theta)\, \Delta_0^{-5} \\
&= a_2^{-1}\{-\alpha \sin\theta \,.\, a_2^3 \Delta_0^{-3} + 3\alpha(\sin\theta - \tfrac{1}{2}\alpha \sin 2\theta)\, a_2^5 \Delta_0^{-5}\}.
\end{aligned}$$

For the secular terms it is possible to write

$$\cos(M_1 - M_2) = \cos(\theta - \varpi_1 + \varpi_2) = \cos\theta\cos(\varpi_1 - \varpi_2)$$
$$\sin(M_1 - M_2) = \sin(\theta - \varpi_1 + \varpi_2) = \sin\theta\cos(\varpi_1 - \varpi_2)$$

since sine terms and cosine terms must combine separately.

178. The secular terms of the second degree in the eccentricities can now be written down in terms of Laplace's coefficients (§ 147) thus:

$$\begin{aligned}
\Delta^{-1} = &+ \tfrac{1}{4} e_1 e_2 \cos(\varpi_1 - \varpi_2) \,.\, a_2^{-1} \\
&\quad \{\tfrac{1}{2}\alpha(b_{\frac{3}{2}}^0 + b_{\frac{3}{2}}^2) + 3\alpha[\tfrac{3}{2}\alpha b_{\frac{5}{2}}^1 - \tfrac{1}{2}(1+\alpha^2)(b_{\frac{5}{2}}^0 + b_{\frac{5}{2}}^2) + \tfrac{1}{4}\alpha(b_{\frac{5}{2}}^1 + b_{\frac{5}{2}}^3)] \\
&\quad + 2\alpha(b_{\frac{3}{2}}^0 + b_{\frac{3}{2}}^2) - 3\alpha^2(b_{\frac{5}{2}}^1 - b_{\frac{5}{2}}^3) \\
&\quad + \alpha(b_{\frac{3}{2}}^0 - b_{\frac{3}{2}}^2) - 3\alpha^2[\alpha(b_{\frac{5}{2}}^0 - b_{\frac{5}{2}}^2) - \tfrac{1}{2}(b_{\frac{5}{2}}^1 - b_{\frac{5}{2}}^3)] \\
&\quad + \alpha(b_{\frac{3}{2}}^0 - b_{\frac{3}{2}}^2) - 3\alpha[(b_{\frac{5}{2}}^0 - b_{\frac{5}{2}}^2) - \tfrac{1}{2}\alpha(b_{\frac{5}{2}}^1 - b_{\frac{5}{2}}^3)]\} \\
&- \tfrac{1}{8}(e_1^2 + e_2^2) \,.\, a_2^{-1}\{\alpha b_{\frac{3}{2}}^1 + 3\alpha[\tfrac{3}{2}\alpha b_{\frac{5}{2}}^0 - (1+\alpha^2) b_{\frac{5}{2}}^1 + \tfrac{1}{2}\alpha b_{\frac{5}{2}}^2]\} \\
&+ \tfrac{1}{2}(e_1^2 + e_2^2) \,.\, a_2^{-1}\{-\alpha b_{\frac{3}{2}}^1 + \tfrac{3}{2}\alpha^2(b_{\frac{5}{2}}^0 - b_{\frac{5}{2}}^2)\}.
\end{aligned}$$

To simplify this expression the recurrence formulae (4) and (5) of § 148 with $j = 0$ are available:

$$(i - s + 1)\alpha b_s^{i+1} - i(1+\alpha^2) b_s^i + (i + s - 1)\alpha b_s^{i-1} = 0$$
$$(i + s) b_s^i = s(1+\alpha^2) b_{s+1}^i - 2s\alpha b_{s+1}^{i+1}.$$

Thus

$$\begin{aligned}
b_{\frac{3}{2}}^1 &= \tfrac{3}{5}(1+\alpha^2) b_{\frac{5}{2}}^1 - \tfrac{6}{5}\alpha b_{\frac{5}{2}}^2 \\
&= \tfrac{3}{5}(-\tfrac{1}{2}\alpha b_{\frac{5}{2}}^2 + \tfrac{5}{2}\alpha b_{\frac{5}{2}}^0) - \tfrac{6}{5}\alpha b_{\frac{5}{2}}^2 = \tfrac{3}{2}\alpha(b_{\frac{5}{2}}^0 - b_{\frac{5}{2}}^0)
\end{aligned}$$

and the last line of the expression disappears. Again

$$\begin{aligned}
&\tfrac{3}{2}\alpha b_{\frac{5}{2}}^0 - (1+\alpha^2) b_{\frac{5}{2}}^1 + \tfrac{1}{2}\alpha b_{\frac{5}{2}}^2 \\
&= \tfrac{3}{5}\{(1+\alpha^2) b_{\frac{5}{2}}^1 + \tfrac{1}{2}\alpha b_{\frac{5}{2}}^2\} - (1+\alpha^2) b_{\frac{5}{2}}^1 + \tfrac{1}{2}\alpha b_{\frac{5}{2}}^2 \\
&= -\tfrac{2}{5}(1+\alpha^2) b_{\frac{5}{2}}^1 + \tfrac{4}{5}\alpha b_{\frac{5}{2}}^2 = -\tfrac{2}{3} b_{\frac{3}{2}}^1.
\end{aligned}$$

Hence the penultimate line of the expression reduces to

$$+\tfrac{1}{8}(e_1^2 + e_2^2) \,.\, a_2^{-1}\alpha b_{\frac{3}{2}}^1$$

which represents all the terms in e_1^2, e_2^2.

The coefficient of $+\frac{1}{4} e_1 e_2 \cos(\varpi_1 - \varpi_2) a_2^{-1}\alpha$ is

$$\begin{aligned}
&+ \tfrac{9}{2} b_{\frac{3}{2}}^0 + \tfrac{1}{2} b_{\frac{3}{2}}^2 - \tfrac{9}{2}(1+\alpha^2) b_{\frac{5}{2}}^0 + \tfrac{21}{4}\alpha b_{\frac{5}{2}}^1 + \tfrac{3}{2}(1+\alpha^2) b_{\frac{5}{2}}^2 + \tfrac{3}{4}\alpha b_{\frac{5}{2}}^3 \\
&= \tfrac{1}{2} b_{\frac{3}{2}}^2 - \tfrac{15}{4}\alpha b_{\frac{5}{2}}^1 + \tfrac{3}{2}(1+\alpha^2) b_{\frac{5}{2}}^2 + \tfrac{3}{4} b_{\frac{5}{2}}^3 \\
&= \tfrac{1}{2} b_{\frac{3}{2}}^2 - \tfrac{15}{14}[2(1+\alpha^2) b_{\frac{5}{2}}^2 - \tfrac{1}{2}\alpha b_{\frac{5}{2}}^3] + \tfrac{3}{2}[(1+\alpha^2) b_{\frac{5}{2}}^2 + \tfrac{1}{2}\alpha b_{\frac{5}{2}}^3] \\
&= \tfrac{1}{2} b_{\frac{3}{2}}^2 - \tfrac{3}{14}[3(1+\alpha^2) b_{\frac{5}{2}}^2 - 6\alpha b_{\frac{5}{2}}^3] \\
&= \tfrac{1}{2} b_{\frac{3}{2}}^2 - \tfrac{3}{2} b_{\frac{3}{2}}^2 = - b_{\frac{3}{2}}^2
\end{aligned}$$

and the whole of this term is therefore

$$-\tfrac{1}{4}e_1e_2\cos(\varpi_1-\varpi_2)\,.\,a_2^{-1}\alpha b_{\frac{3}{2}}^{2}.$$

Hence the terms of the second degree in the eccentricities and inclinations for two planets give finally

$$[\Delta^{-1}] = a_2^{-2}a_1\left\{\tfrac{1}{8}(e_1^2+e_2^2)\,b_{\frac{3}{2}}^{1} - \tfrac{1}{4}e_1e_2\cos(\varpi_1-\varpi_2)\,b_{\frac{3}{2}}^{2}\right\}$$
$$-\tfrac{1}{8}a_2^{-2}a_1\left\{i_1^2+i_2^2-2i_1i_2\cos(\Omega_1-\Omega_2)\right\}b_{\frac{3}{2}}^{1}.$$

But to this order (that is, neglecting the third order in e, i)

$$\xi_1 = eL^{\frac{1}{2}}\cos\varpi,\quad \eta_1 = -eL^{\frac{1}{2}}\sin\varpi$$
$$\xi_2 = iL^{\frac{1}{2}}\cos\Omega,\quad \eta_2 = -iL^{\frac{1}{2}}\sin\Omega.$$

By translating from one system of variables to the other and taking the sum for each pair of planets, it follows that

$$[-U+U_1'] = \tfrac{1}{8}\Sigma m_i m_j\left\{\left(\frac{\xi^2_{i,1}}{L_i}+\frac{\eta^2_{i,1}}{L_i}+\frac{\xi^2_{j,1}}{L_j}+\frac{\eta^2_{j,1}}{L_j}\right)B_1(a_i,a_j)\right.$$
$$-\frac{2}{L_i^{\frac{1}{2}}L_j^{\frac{1}{2}}}(\xi_{i,1}\xi_{j,1}+\eta_{i,1}\eta_{j,1})\,B_2(a_i,a_j)$$
$$\left.-\left[\frac{\xi^2_{i,2}}{L_i}+\frac{\eta^2_{i,2}}{L_i}+\frac{\xi^2_{j,2}}{L_j}+\frac{\eta^2_{j,2}}{L_j}-\frac{2(\xi_{i,2}\xi_{j,2}+\eta_{i,2}\eta_{j,2})}{L_i^{\frac{1}{2}}L_j^{\frac{1}{2}}}\right]B_1(a_i,a_j)\right\}$$

where

$$B_1(a_i,a_j) = \frac{a_i}{a_j^2}\,b_{\frac{3}{2}}^{1}\left(\frac{a_i}{a_j}\right) = \frac{2}{\pi}\int_0^{\pi}\frac{a_ia_j\cos\theta\,.\,d\theta}{(a_i^2+a_j^2-2a_ia_j\cos\theta)^{\frac{3}{2}}}$$
$$B_2(a_i,a_j) = \frac{a_i}{a_j^2}\,b_{\frac{3}{2}}^{2}\left(\frac{a_i}{a_j}\right) = \frac{2}{\pi}\int_0^{\pi}\frac{a_ia_j\cos 2\theta\,.\,d\theta}{(a_i^2+a_j^2-2a_ia_j\cos\theta)^{\frac{3}{2}}}.$$

The coefficients of Laplace are positive. Therefore the quadratic terms in the oblique variables are a negative definite form. Further, by the recurrence formulae,

$$0 = \tfrac{5}{2}\alpha b_{\frac{3}{2}}^{1} - 2(1+\alpha^2)\,b_{\frac{3}{2}}^{2} + \tfrac{3}{2}\alpha b_{\frac{3}{2}}^{3}$$
$$\tfrac{5}{2}b_{\frac{1}{2}}^{2} = \tfrac{1}{2}(1+\alpha^2)\,b_{\frac{3}{2}}^{2} - \alpha b_{\frac{3}{2}}^{3}.$$

Therefore

$$\tfrac{3}{2}b_{\frac{1}{2}}^{2} = \alpha b_{\frac{3}{2}}^{1} - \tfrac{1}{2}(1+\alpha^2)\,b_{\frac{3}{2}}^{2}.$$

But

$$\tfrac{3}{2}b_{\frac{1}{2}}^{1} = \tfrac{1}{2}(1+\alpha^2)\,b_{\frac{3}{2}}^{1} - \alpha b_{\frac{3}{2}}^{2}$$

and therefore

$$3(b_{\frac{1}{2}}^{1}+b_{\frac{1}{2}}^{2}) = (1+\alpha)^2(b_{\frac{3}{2}}^{1}-b_{\frac{3}{2}}^{2})$$

which shows that

$$b_{\frac{3}{2}}^{1} > b_{\frac{3}{2}}^{2},\quad B_1 > B_2.$$

Hence the quadratic terms in the eccentric variables are a positive definite form.

179. The problem of the small eccentricities and inclinations of the planetary system is now brought within the range of the general theory of small oscillations about a steady state of motion. Indeed a knowledge of the principles of this theory shows at once that the variations in the eccentricities and inclinations are periodic and stable, for this follows from the definite (positive or negative) forms of the quadratic terms.

Since (§ 176)

$$[-U+U_1'] = \Sigma \tfrac{1}{2} A_{i,j}(\xi_{i,1}\xi_{j,1} + \eta_{i,1}\eta_{j,1}) + \Sigma \tfrac{1}{2} B_{i,j}(\xi_{i,2}\xi_{j,2} + \eta_{i,2}\eta_{j,2})$$

the corresponding canonical equations are

$$\frac{d\xi_{i,1}}{dt} = \sum_j A_{i,j}\eta_{j,1}, \qquad \frac{d\eta_{i,1}}{dt} = -\sum_j A_{i,j}\xi_{j,1}$$

$$\frac{d\xi_{i,2}}{dt} = \sum_j B_{i,j}\eta_{j,2}, \qquad \frac{d\eta_{i,2}}{dt} = -\sum_j B_{i,j}\xi_{j,2}$$

forming two distinct sets of linear equations with constant coefficients. The results will clearly be of the same general kind for both, and it is only necessary to consider the eccentric variables.

Let the linear transformations

$$\xi_i = \Sigma a_{i,j} p_j, \quad \eta_i = \Sigma a_{i,j} q_j$$

be orthogonal, so that

$$\Sigma \xi_i^2 = \Sigma p_i^2, \quad \Sigma \eta_i^2 = \Sigma q_i^2$$

$$1 = \sum_i a^2_{i,j}, \qquad 0 = \sum_i a_{i,j} a_{i,k}, \quad (j \neq k).$$

Thus

$$\Sigma \xi_i d\eta_i = \sum_i \sum_j \sum_k a_{i,j} a_{i,k} p_j dq_k = \Sigma p_i dq_i$$

which shows that such a transformation is also canonical. Now let

$$\Sigma A_{i,j} \xi_i \xi_j = \Sigma \alpha_i p_i^2.$$

Then

$$\Sigma A_{i,j}\xi_i\xi_j - \alpha_k \Sigma \xi_i^2 = \Sigma \alpha_i p_i^2 - \alpha_k \Sigma p_i^2$$

is an expression which is independent of p_k. Therefore, product terms being reckoned twice,

$$0 = \sum_i \xi_i \left(\sum_j A_{i,j} \frac{\partial \xi_j}{\partial p_k}\right) - \alpha_k \Sigma \xi_i \frac{\partial \xi_i}{\partial p_k}$$

$$= \sum_i \xi_i (\sum_j A_{i,j} a_{j,k}) - \alpha_k \Sigma \xi_i a_{i,k}.$$

This is an identity, satisfied by all values of ξ_i. Hence

$$\sum_j A_{i,j} a_{j,k} - \alpha_k a_{i,k} = 0$$

and this system of equations, for the values $i = 2, 3, \ldots, n$, gives a consistent solution for $a_{j,k}$, provided α_k is a root of the equation

$$\begin{vmatrix} A_{2,2} - \alpha & A_{2,3} & A_{2,4} & \cdots \\ A_{3,2} & A_{3,3} - \alpha & A_{3,4} & \cdots \\ A_{4,2} & A_{4,3} & A_{4,4} - \alpha & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

This is a symmetrical determinant of familiar type, and it is well known that all its roots are real. For the system of the eight major planets it is of the eighth order. It is most unlikely that the equation would have exactly equal roots in a case like this, nor does it in fact happen. But it is to be observed that the occurrence of repeated roots would alter in no way the essential circumstances. The main point is that the *definite* quadratic form can always be reduced to the form $\Sigma \alpha_i p_i^2$ by a linear transformation to normal coordinates. The effect of repeated roots can be seen when there are three planets. Then $\Sigma \alpha_i p_i^2$ corresponds to an ellipsoid, which is one of revolution when two roots α_i are equal. An arbitrary element enters into the direction cosines of the principal axes, which are the coefficients of the transformation. But this does not affect the form of the result or the stability of the motion. It is not necessary to examine the algebra of the subject further, but so much should be mentioned because from the time of Lagrange to Weierstrass in 1858 it was supposed that the occurrence of repeated roots would result in the appearance of the time outside the periodic functions and would be fatal to stability. It is not so.

180. It has been seen that the orthogonal transformation to normal coordinates is also canonical and that the principal function, as far as the eccentric variables are concerned, takes the form

$$V = \Sigma \tfrac{1}{2} \alpha_i (p_i^2 + q_i^2)$$

where α_i is positive, since V is a positive definite form. The canonical equations therefore become

$$\frac{dp_i}{dt} = \alpha_i q_i, \quad \frac{dq_i}{dt} = -\alpha_i p_i$$

and the solution is

$$p_i = C_i \cos(\alpha_i t + h_i), \quad q_i = -C_i \sin(\alpha_i t + h_i)$$

where C_i, h_i are arbitrary constants. This gives the quadratic integrals

$$p_i^2 + q_i^2 = C_i^2.$$

These results are immediately expressed in terms of the previous variables ξ_i, η_i. Thus

$$\xi_i = \Sigma a_{i,j} p_j = \Sigma a_{i,j} C_j \cos(\alpha_j t + h_j)$$
$$\eta_i = \Sigma a_{i,j} q_j = -\Sigma a_{i,j} C_j \sin(\alpha_j t + h_j)$$

where $a_{i,j}$ are definite constants. When the transformation is reversed,

$$p_j = \Sigma a_{i,j}\xi_i, \quad q_j = \Sigma a_{i,j}\eta_i$$

and the quadratic integrals become

$$(\sum_i a_{i,j}\xi_i)^2 + (\sum_i a_{i,j}\eta_i)^2 = C_j^2.$$

The general solution may also be written, with the degree of approximation adopted,

$$e_i L_i^{\frac{1}{2}} \cos \varpi_i = \sum_j a_{i,j} C_j \cos(\alpha_j t + h_j)$$

$$e_i L_i^{\frac{1}{2}} \sin \varpi_i = \sum_j a_{i,j} C_j \sin(\alpha_j t + h_j)$$

which determine the eccentricities and the motions of the perihelia. The question then arises in every case: has the perihelion a mean motion? In other words, is the motion of perihelion, to use the analogy of the simple pendulum, of the circulating or the oscillating type?

The problem, stated in general terms, is not a simple one. But there is one simple case which will serve to explain what is meant and the necessary condition of which is satisfied more often than not. The preceding equations may be regarded as applying to certain coplanar vectors whose tensors are $e_i L_i^{\frac{1}{2}}$, $a_{i,j} C_j$. From this point of view the one vector is represented as the sum of a set of vectors each rotating uniformly. Let the tensor of one vector of the set exceed in length the sum of the tensors of the rest, and let this vector terminate at the origin, the others forming a chain from the other end. It is then geometrically obvious that the representative point at the end of the chain must share in the circulation round the origin of the predominant vector. The perihelion in this case has a mean motion therefore, and it coincides with that, α_i, associated with the large coefficient. The sense of this mean motion is always direct, since α_i is positive. In the same circumstances e_i cannot vanish, but has a lower positive limit.

The condition is clearly satisfied when there are only two planets, unless the two tensors are equal. In this exceptional case it is evident that the mean motion of a perihelion is the same as that of the resultant of the two vectors and is the arithmetic mean, $\frac{1}{2}(\alpha_2 + \alpha_3)$, between their angular motions.

The eight roots of the fundamental determinant range between the values $0''\cdot616$ and $22''\cdot46$ (Stockwell). These are annual motions, so that the corresponding periods lie between 58,000 and 2,100,000 years. Since they are of this order it is evident that e_i, ϖ_i can be developed in powers of the time and that the lowest terms of such expressions will suffice to represent the changes for several centuries. These are the secular inequalities as commonly understood, and it will be seen that they exhibit the initial changes, apart from those of short period, rather than truly secular effects.

181. These results for the eccentricities and perihelia apply almost without change equally to the inclinations and nodes. But there are two differences to be noted. In the first place the principal function is a negative definite form, which may be written after the transformation to normal coordinates,

$$V = -\tfrac{1}{2}\Sigma\beta_i(p_i^2 + q_i^2)$$

where β_i is positive. In the second place, one β_i is zero, or, in other words, the discriminant or Hessian of V (a quadratic form) vanishes. For the part which involves the oblique variable ξ_i may be written (§ 178)

$$V_1 = -\tfrac{1}{2}\Sigma B_{i,j}(L_i^{-\frac{1}{2}}\xi_i - L_j^{-\frac{1}{2}}\xi_j)^2$$

and therefore

$$\frac{\partial V_1}{\partial \xi_i} = -\sum_j L_i^{-\frac{1}{2}} B_{i,j}(L_i^{-\frac{1}{2}}\xi_i - L_j^{-\frac{1}{2}}\xi_j)$$

$$\frac{\partial^2 V}{\partial \xi_i^2} = -\sum_j L_i^{-1} B_{i,j}, \quad \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} = L_i^{-\frac{1}{2}} L_j^{-\frac{1}{2}} B_{i,j}.$$

If then i is the characteristic of a row and j of a column in the Hessian, and each column is multiplied by the corresponding $L_j^{\frac{1}{2}}$, the sum of each row will vanish. Hence the discriminant is identically zero and $\beta = 0$ is a root of the fundamental equation.

The physical reason for this is easily seen. For the canonical equations become

$$\frac{dp_i}{dt} = -\beta_i q_i, \quad \frac{dq_i}{dt} = \beta_i p_i.$$

Corresponding to the root $\beta_i = 0$,

$$p_i = \Sigma b_{i,j}\xi_j = \text{const.}, \quad q_i = \Sigma b_{i,j}\eta_j = \text{const.}$$

which are two linear integrals. The constants need not be zero, and the inclinations may be finite, while their variations vanish. This in fact is the case when the orbits are all coplanar and inclined to the plane of reference. This explains why the fundamental determinant has a zero root. The other seven negative roots when calculated for the solar system are quite similar in magnitude to the positive roots of the determinant in α.

The general solution of the equations when a finite root is in question is

$$p_i = D_i \cos(\beta_i t + k_i), \quad q_i = D_i \sin(\beta_i t + k_i)$$

giving the quadratic integrals

$$p_i^2 + q_i^2 = (\sum_j b_{j,i}\xi_j)^2 + (\sum_j b_{j,i}\eta_j)^2 = D_i^2.$$

From the general solution it follows that

$$i_i L_i^{\frac{1}{2}} \cos\Omega_i = \xi_i = \Sigma b_{i,j} p_j = \Sigma b_{i,j} D_j \cos(\beta_j t + h_j)$$

$$-i_i L_i^{\frac{1}{2}} \sin\Omega_i = \eta_i = \Sigma b_{i,j} q_j = \Sigma b_{i,j} D_j \sin(\beta_j t + h_j)$$

where $b_{i,j}$ are the definite constants of the transformation to normal coordinates. Owing to the zero root in β, t disappears from one term on the right-hand side of each equation, leaving seven periodic terms and one constant, but the form is undisturbed by this fact.

These equations determine the inclinations and the motions of the nodes. The plane of reference is fixed and arbitrary, except in so far as it lies near the average plane of the orbits. Considered as applying to a set of rotating coplanar vectors, the equations show immediately that if one coefficient on the right exceeds the sum of all the rest (taken positively), the node has a mean motion equal and opposite to that of the corresponding vector, and this mean motion is therefore retrograde. When this simple criterion is satisfied, as it is more often than not, it is also evident that the tensor of the vector $i_i L_i^{\frac{1}{2}}$ cannot vanish and that i_i has a lower limit.

182. The sum of the quadratic integrals gives

$$\Sigma\,(p_i^2 + q_i^2) = \Sigma\,(\xi_i^2 + \eta_i^2) = \text{const.}$$

and this applies separately to the eccentric and to the oblique variables. It follows immediately from the canonical equations of § 179 without any transformation. Now ξ_i, η_i contain the factor L_i, which is $m_i\,(m_1 + m_i)^{\frac{1}{2}}\,\mu_{i-1}\mu_i^{-1}\,a_i^{\frac{1}{2}}$ or to the lowest order in the masses $m_i m_1^{\frac{1}{2}} a_i^{\frac{1}{2}}$. Hence

$$\Sigma m_i a_i^{\frac{1}{2}} e_i^2 = \text{const.}$$

$$\Sigma m_i a_i^{\frac{1}{2}} i_i^2 = \text{const.}$$

or, as the latter is more usually written,

$$\Sigma m_i a_i^{\frac{1}{2}} \tan^2 i_i = \text{const.}$$

for the degree of approximation adopted allows of no discrimination between these forms. The constants being small initially it follows that the orbit of no considerable mass in the system can acquire an indefinitely large eccentricity or inclination at the expense of the others as a result of mutual perturbations. These propositions, due to Laplace, clearly have an importance analogous to that of Poisson on the invariability of the mean distances.

The areal velocity in any orbit is

$$(\mu p)^{\frac{1}{2}} = (m_1 + m_i)^{\frac{1}{2}} a_i^{\frac{1}{2}} \cos\phi_i = G_i.$$

The mass factors being $m_i\mu_{i-1}\mu_i^{-1}$ as in § 170, the components of angular momentum are

$$G_i m_i \mu_{i-1}\mu_i^{-1}\,(\sin i_i \sin\Omega_i,\ -\sin i_i \cos\Omega_i,\ \cos i_i)$$
$$= L_i \cos\phi_i\,(\sin i_i \sin\Omega_i,\ -\sin i_i \cos\Omega_i,\ \cos i_i)$$

when the direction cosines of the normal to the orbit are introduced. These components may be written (§ 174)

$$-\eta_{i,2} L_i^{\frac{1}{2}} \cos^{\frac{1}{2}}\phi_i \cos\tfrac{1}{2} i_i,\quad -\xi_{i,2} L_i^{\frac{1}{2}} \cos^{\frac{1}{2}}\phi_i \cos\tfrac{1}{2} i_i,\quad L_i \cos\phi_i \cos i_i$$

or since

$$\xi^2_{i,1} + \eta^2_{i,1} = 2L_i (1 - \cos\phi_i), \quad \xi^2_{i,2} + \eta^2_{i,2} = 2L_i \cos\phi_i (1 - \cos i_i)$$

they can also be expressed in terms of these quantities. The areal integrals then become

$$-\Sigma\eta_{i,2} \{L_i - \tfrac{1}{2}(\xi^2_{i,1} + \eta^2_{i,1}) - \tfrac{1}{4}(\xi^2_{i,2} + \eta^2_{i,2})\}^{\frac{1}{2}} = \text{const.}$$

$$-\Sigma\xi_{i,2} \{L_i - \tfrac{1}{2}(\xi^2_{i,1} + \eta^2_{i,1}) - \tfrac{1}{4}(\xi^2_{i,2} + \eta^2_{i,2})\}^{\frac{1}{2}} = \text{const.}$$

$$\Sigma \{L_i - \tfrac{1}{2}(\xi^2_{i,1} + \eta^2_{i,1}) - \tfrac{1}{2}(\xi^2_{i,2} + \eta^2_{i,2})\} = \text{const.}$$

If the plane of reference is the invariable plane the first two of these constants are zero. In that case, when there are only two planets, η_2/ξ_2 is the same for both and the nodes coincide, which is the property already noticed in § 169 and referred to as the elimination of the nodes.

These integrals, being satisfied identically, remain true when developed according to order and rank. Thus the third equation gives

$$\frac{d}{dt}\Sigma(\xi^2_{i,1} + \eta^2_{i,1} + \xi^2_{i,2} + \eta^2_{i,2}) = \frac{d}{dt}\Sigma L_i = 0$$

$$\Sigma(\xi^2_{i,1} + \eta^2_{i,1} + \xi^2_{i,2} + \eta^2_{i,2}) = \text{const.}$$

which is the sum of the quadratic integrals both for the eccentric and the oblique variables. For L_i has no terms of zero rank, and the purely periodic terms of the first order are excluded from consideration.

Thus L_i is for the present purpose to be regarded as constant. The neglect of terms of the fourth degree in the disturbing function implies the neglect of the third degree in the variables ξ, η themselves. Hence to the same approximation the first two areal integrals give

$$\Sigma L_i^{\frac{1}{2}}\eta_{i,2} = \text{const.}, \quad \Sigma L_i^{\frac{1}{2}}\xi_{i,2} = \text{const.}$$

These then are the two linear integrals found above for the oblique variables, and their physical meaning is thus explained. The constants are now interpreted (to a factor) as the angular momenta of the system about two rectangular axes in the arbitrary plane of reference. In particular, if the invariable plane of the system is taken as the plane of reference, both the constants will become zero.

183. The interpretation of the equations

$$e_i L_i^{\frac{1}{2}} \genfrac{}{}{0pt}{}{\cos}{\sin} \varpi_i = \sum_j a_{i,j} C_j \genfrac{}{}{0pt}{}{\cos}{\sin} (\alpha_j t + h_j)$$

in a vectorial sense has been seen to give a lower limit of e_i when one of the tensors $|a_{i,j}C_j|$ exceeds the sum of the rest. In all cases similar reasoning shows that

$$e_i L_i^{\frac{1}{2}} < \sum_j |a_{i,j}C_j|$$

gives an upper limit of the eccentricity. Similarly the inequality

$$i_i L_i^{\frac{1}{2}} < \sum_j | b_{i,j} D_j |$$

gives an upper limit of the inclination. The actual limits found in this way by Stockwell are of interest and are therefore reproduced.

	Eccentricity		Inclination	
	Max.	Min.	Max.	Min.
Mercury	0·232	0·121	9°·2	4°·7
Venus	0·071	...	3·3	...
Earth	0·068	...	3·1	...
Mars	0·140	0·018	5·9	...
Jupiter	0·061	0·025	0·5	0·2
Saturn	0·084	0·012	1·0	0·8
Uranus	0·078	0·012	1·1	0·9
Neptune	0·015	0·006	0·8	0·6

The effect of periodic inequalities is ignored, and the inclinations are referred to the invariable plane. Minimum figures are given only when a preponderating term exists.

Since $L_i^{\frac{1}{2}}$ contains $m_i^{\frac{1}{2}}$ as a factor these limits have no value when the mass m_i is very small. To consider this case let an infinitesimal mass m_0 be added to the system. Then for the eccentric variables,

$$[-U + U_1'] = \Sigma \tfrac{1}{2} A_{i,j} (\xi_i \xi_j + \eta_i \eta_j) + \sum_j A_{0,j} (\xi_0 \xi_j + \eta_0 \eta_j) + \tfrac{1}{2} A_{0,0} (\xi_0^2 + \eta_0^2).$$

Inspection of the explicit form in § 178 shows that $A_{i,j}$ is of the order of m_i, any of the masses, assumed comparable, of the finite planets; that $A_{0,j}$ is of the order of $m_0^{\frac{1}{2}} m_i^{\frac{1}{2}}$; and that $A_{0,0}$ is again of the order m_i.

The canonical equations give for the infinitesimal planet

$$\frac{d\xi_0}{dt} = \quad A_{0,0}\eta_0 + \Sigma A_{0,j}\eta_j$$

$$\frac{d\eta_0}{dt} = - A_{0,0}\xi_0 - \Sigma A_{0,j}\xi_j.$$

As the new mass is regarded as infinitesimal, the motion of the finite planets will not be influenced, and the former solution

$$\xi_j = \quad \Sigma a_{j,i} C_i \cos(\alpha_i t + h_i)$$
$$\eta_j = - \Sigma a_{j,i} C_i \sin(\alpha_i t + h_i)$$

holds good. Hence

$$\frac{d\xi_0}{dt} - A_{0,0}\eta_0 = - \sum_{i,j} A_{0,j} a_{j,i} C_i \sin(\alpha_i t + h_i)$$

$$\frac{d\eta_0}{dt} + A_{0,0}\xi_0 = - \sum_{i,j} A_{0,j} a_{j,i} C_i \cos(\alpha_i t + h_i).$$

These are the equations for a natural oscillation, together with a set of forced oscillations, and the solution is

$$\xi_0 = \quad a_0 \cos(A_{0,0}t + h_0) - \Sigma A_{0,j} a_{j,i} C_i (A_{0,0} - \alpha_i)^{-1} \cos(\alpha_i t + h_i)$$

$$\eta_0 = -a_0 \sin(A_{0,0}t + h_0) + \Sigma A_{0,j} a_{j,i} C_i (A_{0,0} - \alpha_i)^{-1} \sin(\alpha_i t + h_i)$$

where a_0, h_0 are arbitrary constants. In general this solution shows that the eccentricity (and a similar form applies to the inclination) of the orbit of the infinitesimal mass will remain small. For ξ_0, η_0 contain $m_0^{\frac{1}{2}}$ as a factor, and $A_{0,j}(A_{0,0} - \alpha_i)^{-1}$ is of the order of $m_0^{\frac{1}{2}} m_i^{-\frac{1}{2}}$. An exception occurs when $A_{0,0}$ is nearly equal to α_i, that is, when the period of the free oscillation nearly agrees with one of the forced periods imposed by the main planetary system. The corresponding amplitude then tends to become infinite. This condition is fulfilled at the mean distance from the Sun 1·95, or near the inner limit of the minor planets (Eros excepted), but for the inclinations only ($A_{0,0} = \beta_i$). But before any positive conclusion can be drawn for this case, the extremely limited development of the disturbing function must be remembered*.

* Cf. Charlier's *Mechanik des Himmels*, I.

CHAPTER XVII

SECULAR INEQUALITIES. METHOD OF GAUSS

184. A beautiful method of calculating the secular perturbations of the first order, due to the action of one planet on another, was proposed by Gauss in 1818. It was this method which was applied by Adams to the path of the Leonid meteors. Further developments have been given by several writers, and references will be found in an article* by H. v. Zeipel.

The principle of the method is extremely simple. Equations for the variations of the elements have been found in a suitable form in § 142. As an example we may take ($\mu = n^2a^3$)

$$\frac{di}{dt} = \frac{1}{na^2} \cdot \frac{rW\cos u}{\cos\phi}.$$

Here the right-hand side can be developed in terms of M, M', the mean anomalies of the disturbed and disturbing planets, in the form

$$\frac{di}{dt} = A_{0,0} + \Sigma A_{j,j'}\cos(jM + j'M' + q)$$

and hence, the coefficients being constant in the first approximation,

$$i - i_0 = A_{0,0}t + \Sigma A_{j,j'}\sin(jM + j'M' + q)/(jn + j'n').$$

If therefore the mean motions n, n' are incommensurable, so that $(jn + j'n')$ can never vanish, $A_{0,0}t$ constitutes the secular inequality in i. Now

$$A_{0,0} = \left[\frac{di}{dt}\right]_{0,0} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{di}{dt}\,dM\,dM'$$

$$= \frac{1}{2\pi\, na^2\cos\phi}\int_0^{2\pi} r\cos u\left[\frac{1}{2\pi}\int_0^{2\pi} W\,dM'\right]dM \quad \ldots(1)$$

The component W contains as a factor $k^2m' = n^2a^3m'/(1+m)$. We therefore write

$$\frac{n^2a^3m'}{1+m}W_0 = \frac{1}{2\pi}\int_0^{2\pi} W\,dM'$$

with similar reduced mean values S_0, T_0 corresponding to S, T. If then a series of values of S_0, T_0, W_0 can be calculated for a number of points

* *Encyklopädie d. math. Wiss.*, VI. 2, p. 632.

regularly distributed round the disturbed orbit, they can be introduced into the equations for the variations and a simple quadrature will give the secular perturbations of the several elements, that of a being zero.

185. In calculating S_0, T_0, W_0, the disturbed planet occupies a given fixed point P in its orbit. It is clear that S_0, T_0, W_0 are components of the mean attraction, with respect to the time, exercised at P by a unit mass describing the disturbing orbit, with unit constant of gravitation. They are the same as would result if the disturbing orbit were permanently loaded so as to constitute a material ring of the same total mass, when the density is proportional to dM'/ds'. Thus it is necessary to calculate the attraction of an elliptic ring of this kind.

Let any system of rectangular axes xyz be taken, with origin at P. Let (x_0, y_0, z_0) be the coordinates of the Sun, (x', y', z') those of a point P' on the disturbing orbit, and let $d\sigma'$ be the area of an elementary focal sector, dV' the volume of the tetrahedron on the base $d\sigma'$ with its apex at P. Then

$$2p \,.\, d\sigma' = 6dV' = x_0(y'dz' - z'dy') + y_0(z'dx' - x'dz') + z_0(x'dy' - y'dx')$$

where p is the perpendicular from P on the plane of $d\sigma'$. Hence one component of the required attraction at P is

$$P_x = \frac{1}{2\pi}\int_0^{2\pi}\frac{x'}{\Delta^3}\,dM' = \frac{1}{\pi a'b'}\int\frac{x'}{\Delta^3}\,d\sigma' = \frac{3}{\pi a'b'p}\int\frac{x'}{\Delta^3}\,dV'$$

where a', b' are the semi-axes of the disturbing orbit and $\Delta^2 = x'^2 + y'^2 + z'^2$. This takes account of the first (principal) part of the disturbing function only: the second (indirect) part has been left out of consideration because (§ 159) it gives rise to no secular terms in the perturbations of the first order. It is now to be observed that $x'\Delta^{-3}dV'$ is a homogeneous function of degree 0 in x', y', z', and can therefore be expressed, since $z'dy' - y'dz' = z'^2 d\,(y'/z'), \ldots,$ in terms of x'/z', y'/z', which are connected by some relation

$$f(x'/z',\quad y'/z') = 0$$

which is the equation of the cone having its apex at P and the attracting ring as its section. Thus the integral factor of P_x (and similarly of P_y, P_z) depends only on the form of the cone and not on the particular section. This is true whatever the shape of the ring may be. But in the present case the cone is of the second degree, and the axes may now be identified with its principal axes, $P\,(X, Y, Z)$. Let PZ be the internal axis and α, β the semi-axes of the section $Z = 1$. The coordinates of P' can be written

$$X' = \alpha\cos\tau, \quad Y' = \beta\sin\tau, \quad Z' = 1$$

where τ is the eccentric angle in the section, and

$$\Delta^2 = 1 + \alpha^2\cos^2\tau + \beta^2\sin^2\tau, \quad 6dV' = (-\beta X_0\cos\tau - \alpha Y_0\sin\tau + \alpha\beta Z_0)\,d\tau.$$

Hence

$$P_X = \frac{1}{2\pi a'b'p}\int_0^{2\pi} \frac{\alpha\cos\tau\,(-\beta X_0\cos\tau - \alpha Y_0\sin\tau + \alpha\beta Z_0)\,d\tau}{(1+\alpha^2\cos^2\tau+\beta^2\sin^2\tau)^{\frac{3}{2}}}$$

$$= \frac{-2\alpha\beta X_0}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\tau\,d\tau}{\Delta^3}$$

and similarly

$$P_Y = \frac{-2\alpha\beta Y_0}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\tau\,d\tau}{\Delta^3},\quad P_Z = \frac{2\alpha\beta Z_0}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{d\tau}{\Delta^3}.$$

These components can now be expressed in terms of the complete elliptic integrals

$$F = \int_0^{\frac{1}{2}\pi}\frac{d\tau}{\sqrt{(1-k^2\sin^2\tau)}},\quad E = \int_0^{\frac{1}{2}\pi}\sqrt{(1-k^2\sin^2\tau)}\,d\tau.$$

For, since

$$\frac{d}{d\tau}\cdot\frac{\sin\tau\cos\tau}{\sqrt{(1-k^2\sin^2\tau)}} = \frac{\cos^2\tau - \sin^2\tau + k^2\sin^4\tau}{(1-k^2\sin^2\tau)^{\frac{3}{2}}}$$

$$0 = \int_0^{\frac{1}{2}\pi}\frac{\cos^2\tau\,d\tau}{(1-k^2\sin^2\tau)^{\frac{3}{2}}} - \frac{1}{k^2}(F-E) = \frac{1}{k^2}E - \frac{1-k^2}{k^2}\int_0^{\frac{1}{2}\pi}\frac{d\tau}{(1-k^2\sin^2\tau)^{\frac{3}{2}}}$$

$$= \int_0^{\frac{1}{2}\pi}\frac{\sin^2\tau\,d\tau}{(1-k^2\sin^2\tau)^{\frac{3}{2}}} + \frac{1}{k^2}F - \frac{1}{k^2(1-k^2)}E.$$

Hence

$$P_X = \frac{-2X_0}{\pi a'b'p}\cdot\frac{\alpha\beta}{(\alpha^2-\beta^2)\sqrt{(1+\alpha^2)}}(F-E)$$

$$P_Y = \frac{-2Y_0}{\pi a'b'p}\cdot\frac{\alpha\beta}{(\alpha^2-\beta^2)\sqrt{(1+\alpha^2)}}\left[\frac{1+\alpha^2}{1+\beta^2}E - F\right]$$

$$P_Z = \frac{2Z_0}{\pi a'b'p}\cdot\frac{\alpha\beta}{(1+\beta^2)\sqrt{(1+\alpha^2)}}E$$

where the modulus k of E and F is given by

$$k^2 = \frac{\alpha^2-\beta^2}{1+\alpha^2},\quad 1-k^2 = \frac{1+\beta^2}{1+\alpha^2}.$$

186. It is now necessary to consider the geometry of the problem. Let the angular elements of the disturbed orbit be Ω, i, ω, and of the disturbing orbit Ω', i', ω'. These are referred to the ecliptic, which it is convenient to eliminate by referring the latter orbit directly to the former. With some change in the notation of § 67 the equations there found give

$$\sin\tfrac{1}{2}(\Omega'' + \omega' - \omega'')\sin\tfrac{1}{2}i'' = \sin\tfrac{1}{2}(\Omega' - \Omega)\sin\tfrac{1}{2}(i'+i)$$

$$\cos\tfrac{1}{2}(\Omega'' + \omega' - \omega'')\sin\tfrac{1}{2}i'' = \cos\tfrac{1}{2}(\Omega' - \Omega)\sin\tfrac{1}{2}(i'-i)$$

$$\sin\tfrac{1}{2}(\Omega'' - \omega' + \omega'')\cos\tfrac{1}{2}i'' = \sin\tfrac{1}{2}(\Omega' - \Omega)\cos\tfrac{1}{2}(i'+i)$$

$$\cos\tfrac{1}{2}(\Omega'' - \omega' + \omega'')\cos\tfrac{1}{2}i'' = \cos\tfrac{1}{2}(\Omega' - \Omega)\cos\tfrac{1}{2}(i'-i).$$

Here Ω'' is the distance of the intersection of the two orbits from the ecliptic node of the disturbed orbit, i'' is the mutual inclination of the two planes, and ω'' is the distance of the perihelion of the disturbing orbit from the intersection.

Two sets of rectangular axes, with an arbitrary origin O, are now to be defined. For $O(\xi, \eta, \zeta)$ the directions are those of S, T, W, so that $O\xi$ is parallel to the radius vector at P, $O\eta$ is parallel to the plane of the disturbed orbit and 90° in advance of $O\xi$, and $O\zeta$ is in the direction of the N. pole of this orbit. For the second set, $O(x, y, z)$, Ox is directed towards the perihelion of the disturbing planet, Oy is parallel to the plane of the disturbing orbit and 90° in advance of Ox, and Oz is directed towards the N. pole of this orbit. Let v be the true anomaly at P, and

$$\omega + v - \Omega'' = v_1$$

the distance of P from the intersection of the orbits. Then the relations between the two systems of coordinates are given by the scheme:

	ξ	η	ζ
x	$\cos\omega''\cos v_1+\sin\omega''\sin v_1\cos i''$	$-\cos\omega''\sin v_1+\sin\omega''\cos v_1\cos i''$	$\sin\omega''\sin i''$
y	$-\sin\omega''\cos v_1+\cos\omega''\sin v_1\cos i''$	$\sin\omega''\sin v_1+\cos\omega''\cos v_1\cos i''$	$\cos\omega''\sin i''$
z	$-\sin v_1\sin i''$	$-\cos v_1\sin i''$	$\cos i''$

Thus if r is the radius vector at P, and the origin O be taken at the centre of the disturbing orbit, the coordinates of P are (x_1, y_1, z_1), where

$$x_1 = a'e' + r(\cos\omega''\cos v_1 + \sin\omega''\sin v_1\cos i'')$$

$$y_1 = r(-\sin\omega''\cos v_1 + \cos\omega''\sin v_1\cos i''), \quad z_1 = -r\sin v_1\sin i'' = p$$

and a', e' are the mean distance and eccentricity of the disturbing orbit.

187. Consider now the confocal system of quadrics of which the disturbing orbit is the focal ellipse

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

The parameters λ_1, λ_2, λ_3 of the three quadrics passing through the point (x_1, y_1, z_1) are given by

$$\frac{x_1^2}{a'^2+\lambda} + \frac{y_1^2}{b'^2+\lambda} + \frac{z_1^2}{\lambda} = 1$$

or as the roots of the cubic

$$\begin{aligned}\lambda^3 - \lambda^2(x_1^2 + y_1^2 + z_1^2 - a'^2 - b'^2)&\\ + \lambda(a'^2b'^2 - a'^2y_1^2 - b'^2x_1^2 - a'^2z_1^2 - b'^2z_1^2) - a'^2b'^2z_1^2 = 0 &\quad \text{......(2)}\end{aligned}$$

Now the axes of any tangent cone to a quadric are the normals to the three confocals which can be drawn through the vertex of the cone, and this remains true in the particular case where the focal ellipse is a section of

the cone. Hence the relations between the sets of coordinates (X, Y, Z) and (x, y, z) are given by the scheme:

$$\begin{array}{cccc} & x & y & z \\ X & p_1 x_1 (a'^2+\lambda_1)^{-1} & p_1 y_1 (b'^2+\lambda_1)^{-1} & p_1 z_1/\lambda_1 \\ Y & p_2 x_1 (a'^2+\lambda_2)^{-1} & p_2 y_1 (b'^2+\lambda_2)^{-1} & p_2 z_1/\lambda_2 \\ Z & p_3 x_1 (a'^2+\lambda_3)^{-1} & p_3 y_1 (b'^2+\lambda_3)^{-1} & p_3 z_1/\lambda_3 \end{array}$$

where p_1, p_2, p_3 are such that

$$p_1^2 \{x_1^2 (a'^2+\lambda_1)^{-2} + y_1^2 (b'^2+\lambda_1)^{-2} + z_1^2 \lambda_1^{-2}\} = 1, \ldots.$$

When combined with the scheme given above for (x, y, z), (ξ, η, ζ), this gives the relations between (X, Y, Z) and (ξ, η, ζ).

The equation of the cone is

$$\frac{(zx_1 - xz_1)^2}{a'^2} + \frac{(zy_1 - yz_1)^2}{b'^2} = (z - z_1)^2$$

for this is clearly homogeneous and of the second degree in $x - x_1$, $y - y_1$, $z - z_1$, and its section by the plane $z = 0$ is the disturbing orbit. Transposed to parallel axes through its vertex (x_1, y_1, z_1) it becomes

$$-\frac{x^2}{a'^2} - \frac{y^2}{b'^2} - \frac{z^2}{z_1^2}\left(\frac{x_1^2}{a'^2} + \frac{y_1^2}{b'^2} - 1\right) + \frac{2yz}{b'^2} \cdot \frac{y_1}{z_1} + \frac{2zx}{a'^2} \cdot \frac{x_1}{z_1}$$

$$\equiv X^2/\lambda_1 + Y^2/\lambda_2 + Z^2/\lambda_3 = F_{-1} = 0.$$

The justification for identifying these two forms is seen on comparing the three functions of the coefficients which remain invariant under a rotation of the axes. It will then be found that the results are equivalent to the relations between the coefficients and roots of (2).

It is convenient to write down the equation of the reciprocal cone. The coefficients are the minors of the discriminant of the previous equation $F_{-1} = 0$. Hence with due care in choosing the right multiplier the desired equation may be written

$$x^2 (x_1^2 - a'^2) + y^2 (y_1^2 - b'^2) + z^2 z_1^2 + 2yz\, y_1 z_1 + 2zx\, z_1 x_1 + 2xy\, x_1 y_1$$

$$\equiv \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = F_1 = 0$$

the invariant relations being identical with those between the coefficients and roots of (2).

Also

$$x^2 + y^2 + z^2 \equiv X^2 + Y^2 + Z^2 = \xi^2 + \eta^2 + \zeta^2 = F_0$$

and it is evident that F_{-1}, F_1 can also be readily expressed, by means of the transformation scheme of § 186, in terms of ξ, η, ζ.

188. Two of the roots of the cubic (2) are negative and one positive, since two of the corresponding quadrics are hyperboloids and one an ellipsoid. Let

$$\lambda_1 < \lambda_2 < 0 < \lambda_3.$$

The axis of Z is then the internal axis of the cone F_{-1} and it follows that

$$\alpha^2 = -\frac{\lambda_1}{\lambda_3}, \quad \beta^2 = -\frac{\lambda_2}{\lambda_3}, \quad k^2 = \frac{\alpha^2 - \beta^2}{1 + \alpha^2} = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}.$$

The elliptic integrals F, E can therefore be found. The coordinates of the Sun relative to the point P are $x_0 = a'e' - x_1$, $y_0 = -y_1$, $z_0 = -z_1$ in the system (x, y, z) and (X_0, Y_0, Z_0) can be deduced by the transformation scheme of § 187. Hence P_X, P_Y, P_Z become known, and the components $P_\xi = S_0$, $P_\eta = T_0$, $P_\zeta = W_0$ may be derived by applying the two transformations of §§ 186 and 187.

It is unnecessary here to consider the equations for all the inequalities. As a type, (1) now becomes

$$\left(\frac{di}{dt}\right)_{0,0} = \frac{nam'}{(1+m)\cos\phi} \cdot \frac{1}{2\pi}\int_0^{2\pi} r\cos u \,.\, W_0 dM.$$

Suppose that j values ψ_s of $\psi = r\cos u \,.\, W_0$ have been found, corresponding to j points around the disturbed orbit at which M has equidistant values, $0, 2\pi/j, \ldots, 2(j-1)\pi/j$. Then (Chapter XXIV)

$$\psi = a_0 + \Sigma a_i \cos iM + \Sigma b_i \sin iM$$

where

$$a_0 = \frac{1}{j}\sum_s \psi_s, \quad a_i = \frac{2}{j}\sum_s \psi_s \cos\frac{2si\pi}{j}, \quad b_i = \frac{2}{j}\sum_s \psi_s \sin\frac{2si\pi}{j}.$$

Hence

$$\left(\frac{di}{dt}\right)_{0,0} = \frac{nam'}{(1+m)\cos\phi} \,.\, a_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and it is only necessary to calculate the average value of ψ_s to have the secular inequality. For the major planets $j = 12$ practically suffices. The summation formula for a_0 really gives $a_0 + a_j + \ldots$. It is therefore necessary to take j large enough to make a_j negligible. The number of points to be taken on the disturbed orbit thus depends on the practical convergency of the series $a_0, a_1, a_2, \ldots$.

It is, however, preferred to take points equidistant in E, the eccentric anomaly, instead of M, since this secures a more even distribution in arc. The advantage of this course seems scarcely obvious, because it appears to weight unduly the part of the orbit which is passed over rapidly. But the modification is easily made. In this case

$$\psi = a_0 + \Sigma a_i \cos iE + \Sigma b_i \sin iE$$

where again

$$a_0 = \frac{1}{j}\sum_s \psi_s, \quad a_i = \frac{2}{j}\sum_s \psi_s \cos\frac{2si\pi}{j}, \quad b_i = \frac{2}{j}\sum_s \sin\frac{2si\pi}{j}$$

but the meaning of ψ will be changed. For

$$dM = (1 - e\cos E)\, dE = a^{-1}r\,.\,dE$$

and (1) may be written

$$\left(\frac{di}{dt}\right)_{0,0} = \frac{nam'}{(1+m)\cos\phi}\cdot\frac{1}{2\pi}\int_0^{2\pi} a^{-1}r^2\cos u\,.\,W_0 dE.$$

Hence (3) will still hold good if a_0 is the simple mean value of ψ, where ψ is now $a^{-1}r^2\cos u\,.\,W_0$.

189. The cubic (2) has three real roots and can be easily solved. It is now to be seen that the solution can be avoided. Let the equation be written

$$\lambda^3 + 3k_1\lambda^2 + 3k_2\lambda + k_3 = 0$$

the roots being λ_1, λ_2, λ_3, and let the result of removing the second term be

$$4\lambda'^3 - g_2\lambda' - g_3 = 0$$

of which the roots are e_1, e_2, e_3. Then

$$g_2 = -4(e_2e_3 + e_3e_1 + e_1e_2) = 12(k_1^2 - k_2)$$

$$g_3 = 4e_1e_2e_3 = -4(2k_1^3 - 3k_1k_2 + k_3)$$

and

$$3e_1 = 2\lambda_1 - \lambda_2 - \lambda_3, \quad 3e_2 = 2\lambda_2 - \lambda_3 - \lambda_1, \quad 3e_3 = 2\lambda_3 - \lambda_1 - \lambda_2$$

$$e_1 < e_2 < e_3, \quad e_1 + e_2 + e_3 = 0.$$

Thus

$$\Delta^2 = 1 + \alpha^2\cos^2\tau + \beta^2\sin^2\tau = \lambda_3^{-1}\{(\lambda_3 - \lambda_1)\cos^2\tau + (\lambda_3 - \lambda_2)\sin^2\tau\}$$
$$= \lambda_3^{-1}\{(e_3 - e_1)\cos^2\tau + (e_3 - e_2)\sin^2\tau\} = \lambda_3^{-1}\Delta'^2$$

and the components to be calculated are

$$P_X = \frac{-2X_0(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\tau\, d\tau}{\Delta'^3}, \quad P_Y = \frac{-2Y_0(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\tau\, d\tau}{\Delta'^3},$$

$$P_Z = \frac{2Z_0(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}}{\pi a'b'p}\int_0^{\frac{1}{2}\pi}\frac{d\tau}{\Delta'^3} \quad \text{.........................(4)}$$

where $\lambda_1\lambda_2\lambda_3 = -k_3$. It is clearly possible to write consistently

$$\sin^2\tau = \frac{e_3 - e_1}{e_2 - e_1}\cdot\frac{s - e_2}{s - e_3}, \quad \cos^2\tau = \frac{e_2 - e_3}{e_2 - e_1}\cdot\frac{s - e_1}{s - e_3}, \quad \Delta'^2 = \frac{(e_3 - e_1)(e_2 - e_3)}{s - e_3}$$

whence

$$2\sin\tau\cos\tau\frac{d\tau}{ds} = \frac{(e_3 - e_1)(e_2 - e_2)}{(e_2 - e_1)(s - e_3)^2}$$

and

$$\frac{4}{\Delta'^2}\left(\frac{d\tau}{ds}\right)^2 = \frac{1}{(s - e_1)(s - e_2)(s - e_3)}.$$

But this can be written

$$\Delta'^{-1}\, d\tau = du, \quad \wp(u) = s$$

where $\wp(u)$ is the Weierstrassian elliptic function formed with the roots e_1, e_2, e_3. When $\tau = 0$, $\wp(u) = e_2$, $u = \omega_2$; when $\tau = \frac{1}{2}\pi$, $\wp(u) = e_1$, $u = \omega_1$. Hence

$$\int_0^{\frac{1}{2}\pi} \frac{d\tau}{\Delta'^3} = \int_{\omega_2}^{\omega_1} \frac{\wp(u) - e_3}{(e_3 - e_1)(e_2 - e_3)}\, du = \left[\frac{\zeta(u) + e_3 u}{(e_3 - e_1)(e_2 - e_3)}\right]_{\omega_1}^{\omega_2} = \frac{\eta + e_3\omega}{(e_3 - e_1)(e_2 - e_3)}$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\tau\, d\tau}{\Delta'^3} = \int_{\omega_2}^{\omega_1} \frac{\wp(u) - e_2}{(e_2 - e_1)(e_2 - e_3)}\, du = \left[\frac{\zeta(u) + e_2 u}{(e_2 - e_1)(e_2 - e_3)}\right]_{\omega_1}^{\omega_2} = \frac{\eta + e_2\omega}{(e_2 - e_1)(e_2 - e_3)}$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^2\tau\, d\tau}{\Delta'^3} = \int_{\omega_2}^{\omega_1} \frac{\wp(u) - e_1}{(e_2 - e_1)(e_3 - e_1)}\, du = \left[\frac{\zeta(u) + e_1 u}{(e_2 - e_1)(e_3 - e_1)}\right]_{\omega_1}^{\omega_2} = \frac{\eta + e_1\omega}{(e_2 - e_1)(e_3 - e_1)}$$

where

$$\eta = \zeta(\omega_2) - \zeta(\omega_1), \quad \omega = \omega_2 - \omega_1.$$

The quantities ω and η will now be found.

190. The reader who is unacquainted with the theory of elliptic functions will notice that nothing beyond the definitions of the functions $\wp(u)$, $\zeta(u)$ is here involved, and that these can be easily inferred. In fact, if the variable s be retained, it is easily seen that

$$\omega = \int_{e_1}^{e_2} \frac{ds}{\sqrt{\{4(s - e_1)(s - e_2)(s - e_3)\}}}, \quad \eta = -\int_{e_1}^{e_2} \frac{s\,ds}{\sqrt{\{4(s - e_1)(s - e_2)(s - e_3)\}}}$$

where

$$4(s - e_1)(s - e_2)(s - e_3) = 4s^3 - g_2 s - g_3, \quad e_1 < e_2 < e_3.$$

The range of integration is the finite interval between the roots in which the integrals are real. Let

$$s = (\tfrac{1}{3}g_2)^{\frac{1}{2}} \cos\theta, \quad \cos 3\gamma = (27 g_3^2 g_2^{-3})^{\frac{1}{2}} = g^{-\frac{1}{2}}.$$

The values of θ corresponding to e_1, e_2, e_3 in order are clearly

$$\theta_1 = \tfrac{2}{3}\pi + \gamma, \quad \theta_2 = \tfrac{2}{3}\pi - \gamma, \quad \theta_3 = \gamma < \tfrac{1}{3}\pi$$

since

$$4s^3 - g_2 s - g_3 = (\tfrac{1}{3}g_2)^{\frac{3}{2}} (\cos 3\theta - \cos 3\gamma).$$

Hence

$$\omega = (\tfrac{1}{3}g_2)^{-\frac{1}{4}} \int_{\theta_2}^{\theta_1} \frac{\sin\theta\, d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}}, \quad \eta = -\tfrac{1}{2}(\tfrac{1}{3}g_2)^{\frac{1}{4}} \int_{\theta_2}^{\theta_1} \frac{\sin 2\theta\, d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}}.$$

Now the Mehler-Dirichlet integral* gives

$$P_n(\cos 3\gamma) = \frac{1}{\pi} \int_{-3\gamma}^{3\gamma} \frac{e^{(n+\frac{1}{2})\iota\phi}\, d\phi}{\sqrt{(2\cos\phi - 2\cos 3\gamma)}}$$

where P_n denotes Legendre's function of the first kind and order n. Let $\phi = 3\theta - 2\pi$, and then

$$\int_{\theta_2}^{\theta_1} \frac{e^{3(n+\frac{1}{2})\iota\theta}\, d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}} = \tfrac{1}{3}\sqrt{2}\,\pi e^{(2n+1)\iota\pi} P_n(\cos 3\gamma)$$

* Cf. Whittaker's *Modern Analysis*, p. 219; Whittaker and Watson, p. 308.

whence

$$\int_{\theta_2}^{\theta_1} \frac{\sin 3(n+\frac{1}{2})\,\theta\,d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}} = \tfrac{1}{3}\sqrt{2}\pi \sin(2n+1)\pi P_n(\cos 3\gamma).$$

Now put $n = -\frac{1}{6}$ and $+\frac{1}{6}$ in succession. Thus

$$\int_{\theta_2}^{\theta_1} \frac{\sin\theta\,d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}} = 6^{-\frac{1}{2}}\pi P_{-\frac{1}{6}}(\cos 3\gamma)$$

$$\int_{\theta_2}^{\theta_1} \frac{\sin 2\theta\,d\theta}{\sqrt{(\cos 3\theta - \cos 3\gamma)}} = -6^{-\frac{1}{2}}\pi P_{\frac{1}{6}}(\cos 3\gamma).$$

But the Legendre's functions can be expressed in the form of hypergeometric series* $F(-n, n+1, 1, \sin^2\frac{3}{2}\gamma)$. Hence finally

$$\omega = \pi(12g_2)^{-\frac{1}{4}} F(\tfrac{1}{6}, \tfrac{5}{6}, 1, \sin^2\tfrac{3}{2}\gamma)$$

$$\eta = \tfrac{1}{12}\pi(12g_2)^{\frac{1}{4}} F(-\tfrac{1}{6}, \tfrac{7}{6}, 1, \sin^2\tfrac{3}{2}\gamma)$$

where $\sin^2\frac{3}{2}\gamma = \frac{1}{2}(1 - g^{-\frac{1}{2}})$. Thus ω and η are expressed in a form not requiring the solution of the cubic equation.

These hypergeometric series are not the same as those originally found by H. Bruns as the solution of the problem. But the latter are easily deduced. For $P_n(z)$ satisfies the differential equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)y = 0.$$

The result of changing the independent variable to $x = 1 - z^2$ is

$$x(x-1)\frac{d^2y}{dx^2} + (\tfrac{3}{2}x - 1)\frac{dy}{dx} - \tfrac{1}{4}n(n+1)y = 0$$

which is satisfied by the hypergeometric series $F(-\frac{1}{2}n, \frac{1}{2}n + \frac{1}{2}, 1, x)$. When $z = \cos 3\gamma$, $x = \sin^2 3\gamma = g^{-1}(g-1)$ and since there can be only one convergent series for y in powers of x, this is it. The above series may therefore be replaced by

$$F(\tfrac{1}{12}, \tfrac{5}{12}, 1, \sin^2 3\gamma), \quad F(-\tfrac{1}{12}, \tfrac{7}{12}, 1, \sin^2 3\gamma)$$

which are the series obtained by Bruns.

191. Let the origin of coordinates now be taken at the Sun, the point P being at (X, Y, Z) or $(-X_0, -Y_0, -Z_0)$. Then the components P_X, P_Y, P_Z (4) can be derived by partial differentiation from the potential

$$V = \frac{(-k_3)^{\frac{1}{2}}}{\pi a'b'p}\int_0^{\frac{1}{2}\pi} \frac{X^2\cos^2\tau + Y^2\sin^2\tau - Z^2\,d\tau}{\Delta'^3}$$

$$= \frac{(-k_3)^{\frac{1}{2}}}{\pi a'b'p}\cdot\frac{\eta G_1 + \omega G_2}{(e_3-e_2)(e_3-e_1)(e_2-e_1)}$$

* Cf. Whittaker's *Modern Analysis*, p. 214; Whittaker and Watson, p. 305.

where

$$G_1 = (e_3 - e_2) X^2 + (e_1 - e_3) Y^2 + (e_2 - e_1) Z^2$$

$$G_2 = e_1 (e_3 - e_2) X^2 + e_2 (e_1 - e_3) Y^2 + e_3 (e_2 - e_1) Z^2.$$

Now by ordinary multiplication of determinants

$$\begin{vmatrix} X^2 & Y^2 & Z^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \end{vmatrix} = \begin{vmatrix} F_1 & F_0 & F_{-1} \\ \Sigma\lambda_1^2 & \Sigma\lambda_1 & 3 \\ \Sigma\lambda_1 & 3 & \Sigma\lambda_1^{-1} \end{vmatrix}$$

and

$$\begin{vmatrix} X^2 & Y^2 & Z^2 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \end{vmatrix} = \begin{vmatrix} F_1 & F_0 & F_{-1} \\ 3 & \Sigma\lambda_1^{-1} & \Sigma\lambda_1^{-2} \\ \Sigma\lambda_1 & 3 & \Sigma\lambda_1^{-1} \end{vmatrix}$$

where

$$\lambda^3 + 3k_1\lambda^2 + 3k_2\lambda + k_3 = 0$$

$$4\lambda'^3 - g_2\lambda' - g_3 = 0, \quad \lambda + k_1 = \lambda'$$

and e_1, e_2, e_3 are the roots in λ' corresponding to λ_1, λ_2, λ_3. The first determinant is clearly $-G_1$ and the determinant below it is

$$\Sigma X^2 (\lambda_2^{-1} - \lambda_3^{-1}) = -k_3^{-1} \Sigma\lambda_1 (\lambda_3 - \lambda_2) X^2 = -k_3^{-1} (G_2 - k_1 G_1).$$

The multiplying determinant in both identities is

$$-(\lambda_1\lambda_2\lambda_3)^{-1} (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1) = \tfrac{1}{4} k_3^{-1} (g_2^3 - 27g_3^2)^{\frac{1}{2}}$$

and the determinants on the right-hand side are easily expressed in terms of k_1, k_2, k_3. They are respectively $9k_3^{-1}H_1$ and $-9k_3^{-2}H_2$, where

$$H_1 = F_1 (k_1k_2 - k_3) + F_0 (3k_1^2k_2 - 2k_2^2 - k_1k_3) + 2F_{-1} (k_1^2 - k_2) k_3$$

and

$$H_2 = 2F_1 (k_2^2 - k_1k_3) + F_0 (3k_1k_2^2 - 2k_1^2k_3 - k_2k_3) + F_{-1} (k_1k_2 - k_3) k_3.$$

Hence

$$V = \frac{144 (-k_3)^{\frac{1}{2}}}{\pi a'b'p} \cdot \frac{H_2\omega - H_1 (\eta + k_1\omega)}{g_2^3 - 27g_3^2} \quad \text{..................(5)}$$

But F_1, F_0, F_{-1} have been expressed (§ 187) in terms of (x, y, z). Hence the system of coordinates (X, Y, Z) has been completely eliminated from the problem.

192. Now V is a homogeneous quadratic function in (x, y, z) and can be reduced to the same form in (ξ, η, ζ). But its complete expression is not required, because S_0, T_0, W_0 are its partial differential coefficients at the point P $(r, 0, 0)$. It is therefore

$$V = (S_0\xi + 2T_0\eta + 2W_0\zeta)\,\xi/2r + \dots \quad \text{..................(6)}$$

and the terms which do not contain ξ can be neglected. Thus F_0 is ξ^2 simply. Let the transformation scheme of § 186 be written

$$\begin{aligned} x &= l_1\xi + m_1\eta + n_1\zeta, & x_1 &= l_1 r + a'e' \\ y &= l_2\xi + m_2\eta + n_2\zeta, & y_1 &= l_2 r \\ z &= l_3\xi + m_3\eta + n_3\zeta, & z_1 &= l_3 r \end{aligned}$$

with the usual relations of an orthogonal substitution. Then

$$\begin{aligned} F_1 &= (xx_1 + yy_1 + zz_1)^2 - (a'^2x^2 + b'^2y^2) \\ &= (a'e'x + r\xi)^2 - (a'^2x^2 + b'^2y^2) \\ &= r^2\xi^2 + 2a'e'r\xi x - b'^2F_0 + b'^2z^2 \\ &= \xi\{\xi(r^2 - b'^2 + b'^2l_3^2 + 2a'e'rl_1) + 2\eta(a'e'rm_1 + b'^2l_3m_3) + 2\zeta(a'e'rn_1 + b'^2l_3n_3)\} \end{aligned}$$

with neglect of terms not containing ξ. Similarly

$$F_{-1} = z^2/z_1^2 - (zx_1 - xz_1)^2/a'^2z_1^2 - (zy_1 - yz_1)^2/b'^2z_1^2.$$

The last term does not contain ξ and hence

$$\begin{aligned} a'^2r^2l_3^2F_{-1} &= a'^2(l_3\xi + m_3\eta + n_3\zeta)^2 - \{a'e'z + r\eta(l_1m_3 - l_3m_1) + r\zeta(l_1n_3 - l_3n_1)\}^2 \\ &= b'^2(l_3\xi + m_3\eta + n_3\zeta)^2 - 2a'e'rl_3\xi(-n_2\eta + m_2\zeta) \end{aligned}$$

or

$$F_{-1} = \{b'^2l_3\xi + 2\eta(b'^2m_3 + a'e'rn_2) + 2\zeta(b'^2n_3 - a'e'rm_2)\}\,\xi/a'^2r^2l_3.$$

Thus F_1, F_0, F_{-1} are now expressed, as far as necessary, in terms of ξ, η, ζ. It remains to calculate H_1 and H_2, and then the simple comparison of the coefficients of ξ^2, $\xi\eta$, $\xi\zeta$ in (5) and (6) gives S_0, T_0, W_0.

It must be understood that it is not the object here to obtain the most practical form of calculation in its final shape, but rather to explain the mathematical principles involved and to be content with showing how the computation might be carried out. The method was not developed by Gauss in the complete form which is necessary for practical computations. This was done by Hill. The introduction of elliptic functions in the modern form is due to Halphen.

CHAPTER XVIII

SPECIAL PERTURBATIONS

193. In Chapter XV some explanation has been given of the various classes into which planetary perturbations naturally fall when regarded from a practical point of view. There is, however, another kind of distinction which can be drawn among perturbations, depending on the mode of calculation and expression. When they are expressed in an analytical form, from which their values can be deduced for any time simply by giving t its appropriate value, they are called *absolute* perturbations. For all the major planets a theory has been developed in this form. But such a theory, if it is to be complete and accurate, demands immense labour, which is justified if positions of a planet are constantly required. Moreover questions of general theory must nearly always be based on analytical forms. On the other hand there are bodies which are observed during one short period only, like the majority of comets, or at relatively long intervals, like the periodic comets. In such cases, which include also the orbits of the minor planets, the method of quadratures is resorted to, partly in order to save labour and partly to avoid difficulties which have not hitherto been surmounted by analysis. Perturbations calculated in this way are called *special* perturbations. The advantage of the method is that it is generally applicable, though against this must be set the frequent necessity of continuing the calculation without a break through long intervals when no observations have been made, and the impossibility of making any general inference as to the motion outside the actual period covered by the computations. There are exceptions to this statement, because important researches have been made with success into the origin of comets by the method of special perturbations, and the periodic solutions of the problem of three bodies have also been largely investigated by the method of quadratures. But generally the services of this method have been of a practical rather than a theoretical kind.

The method of quadratures involves an arithmetical technique with which the reader may not be familiar. It therefore lies strictly outside the intended scope of this work, which is not concerned with the actual details of practical calculation. But the computation of special perturbations fills so large a place in the practice of astronomy at the present time that it cannot be dismissed

without some description. Accordingly, in order to interrupt the treatment of dynamical questions as little as possible, a brief account of the algebra of difference tables is given in the final chapter of the book, and the results will be quoted here without proof.

194. Let y_n be a tabulated function of the argument $t = a + nw$, where n represents a series of consecutive integers and w is a constant tabular interval. As the practical formulae of quadrature depend on central differences, it will be convenient to represent the difference table thus:

...	...	...	...	...	...	...	...	...
$K^{-1}y_n$	...	y_n	...	Ky_n	...	K^2y_n	...	...
...	$\Delta K^{-1}y_n$	...	Δy_n	...	ΔKy_n	...	ΔK^2y_n	...
...	...	...	...	...	...	...	...	...

Here y_n is tabulated in a vertical column and the successive differences on the right are formed directly in the usual way. Thus $\Delta y_n = y_{n+1} - y_n$, and the commutative operator K, which is clearly appropriate to central (or horizontal) differences, represents a move two places to the right on a horizontal line of the table. Similarly K^{-1} represents a horizontal move two places to the left. Two columns are shown on the left of the tabulated function, and these are known as the first and second summation columns. The relation of each to the adjacent columns on the right is precisely the same as that holding between any two consecutive difference columns. Thus the first summation column contains the differences of the second, and the differences of the first are the successive values of the function itself. The first column can therefore be based on an arbitrary constant and formed in the downward direction by adding the numerical values of the function successively. The second summation column is based on a second arbitrary constant and formed from the first in the same way.

The table thus constructed has alternate blank spaces. These are now filled by the insertion of the arithmetic means of the entries standing immediately above and below each space. In its completed form the table may be represented thus:

...	...	...	...	...	...	...	...	...
$K^{-1}y_n$	...	y_n	$[ky_n]$	Ky_n	$[kKy_n]$	K^2y_n	$[kK^2y_n]$	...
...	$\Delta K^{-1}y_n$	$[k'y_n]$	Δy_n	$[k'Ky_n]$	ΔKy_n	$[k'K^2y_n]$	ΔK^2y_n	...
...	...	...	...	...	...	...	...	...

where the mean differences are distinguished by k to the *right* of a simple difference or by k' *below* a simple difference. As a matter of fact,

$$k' = 1 + \tfrac{1}{2}\Delta, \quad k = \Delta(1 + \tfrac{1}{2}\Delta)(1 + \Delta)^{-1}, \quad K = \Delta^2(1 + \Delta)^{-1}$$

but for the immediate purpose in view these operators serve merely to define the position of entries in the difference table. They are all algebraic.

195. The formulae available for executing the necessary quadratures can now be given. Numbered as in the last chapter of the book, to which reference can be made for proofs, they are these:

$$w^{-1}\int_c^{a+nw} y\,dt = k\left(K^{-1} - \frac{1}{12} + \frac{11}{720}K - \frac{191}{60480}K^2 + \dots\right)y_n \text{......(28)}$$

$$w^{-1}\int_c^{a+mw} y\,dt = \Delta\left(K^{-1} + \frac{1}{24} - \frac{17}{5760}K + \frac{367}{967680}K^2 - \dots\right)y_n \text{...(26)}$$

$$w^{-2}\int_b^{a+nw}\left[\int_c^x y\,dt\right]dt = \left(K^{-1} + \frac{1}{12} - \frac{1}{240}K + \frac{31}{60480}K^2 - \dots\right)y_n \text{.........(30)}$$

$$w^{-2}\int_b^{a+mw}\left[\int_c^x y\,dt\right]dt = k'\left(K^{-1} - \frac{1}{24} + \frac{17}{1920}K - \frac{367}{193536}K^2 + \dots\right)y_n \text{...(31)}$$

where m is written in the upper limit in the place of $n + \frac{1}{2}$. The commutative operator k must of course be carefully distinguished from the Gaussian constant k.

The lower limits, b and c, are arbitrary and correspond with the arbitrary constants involved in forming the first and second summation columns. If the lower limit is to be $c = a$,

$$\Delta K^{-1}y_0 = \tfrac{1}{2}y_0 + k\left(\frac{1}{12} - \frac{11}{720}K + \frac{191}{60480}K^2 - \dots\right)y_0 \text{.........(29)}$$

which fixes one constituent of the first column, and the rest follow. If the lower limit is to be $c = a + \frac{1}{2}w$,

$$\Delta K^{-1}y_0 = \Delta\left(-\frac{1}{24} + \frac{17}{5760}K - \frac{367}{967680}K^2 + \dots\right)y_0 \text{.........(27)}$$

Similarly, if the lower limit b of the second integration is a,

$$K^{-1}y_0 = \left(-\frac{1}{12} + \frac{1}{240}K - \frac{31}{60480}K^2 + \dots\right)y_0 \text{............(32)}$$

and the value of this particular constituent makes the whole of the second summation column determinate. If the lower limit is $b = a + \frac{1}{2}w$,

$$K^{-1}y_0 = -\tfrac{1}{2}\Delta K^{-1}y_0 + k'\left(\frac{1}{24} - \frac{17}{1920}K + \frac{367}{193536}K^2 - \dots\right)y_0 \text{...(33)}$$

In general, $b = c$ and (29) and (32) are used together, or (27) and (33). In the latter case (33) may also be written

$$K^{-1}y_0 = \left\{\frac{1}{24}(1+\Delta) - \frac{17}{5760}(3+2\Delta)K + \frac{367}{967680}(5+3\Delta)K^2 - \dots\right\}y_0 \text{...(34)}$$

In whatever way the lower limits are determined, (28) and (30) will give the integrals to the upper limit $a + nw$, and (26) and (31) to the upper limit

$$a + (n + \tfrac{1}{2})w.$$

196. The application of quadratures to the solution of differential equations such as arise in dynamical problems can be explained by a simple but fairly general form. Consider the equation

$$\frac{d^2x}{dt^2}=f(x, t)$$

or, as it may be written,

$$D^2x = X.$$

Hence, by (30),

$$\begin{aligned} x &= w^2(wD)^{-2}X \\ &= w^2\left\{K^{-1}+\frac{1}{12}-\frac{1}{240}K+\frac{31}{60480}K^2-\ldots\right\}X \end{aligned}$$

or

$$Kx = w^2\left\{1+\frac{1}{12}K-\frac{1}{240}K^2+\ldots\right\}X \quad\text{.....................(1)}$$

Now suppose that we have a solution in progress, giving at a certain stage,

t_n	x_n		Kx_n	X_n		KX_n		
		$x_{n+1}-x_n$			...		...	
t_{n+1}	x_{n+1}		Kx_{n+1}	X_{n+1}		KX_{n+1}		(K^2X_{n+1})
		$x_{n+2}-x_{n+1}$			...		...	
t_{n+2}	x_{n+2}			X_{n+2}				

Here X_n is a known function of x_n and t_n. It is required to find x_{n+3} and X_{n+3} which depend on t_{n+3} and on one another, so that they cannot be calculated directly. For simplicity the time interval w may be imagined to be so small that $\frac{1}{240}K^2X_{n+1}$ is negligible. The general run of the differences KX will suggest a close guess to the value of KX_{n+2}, though the true value requires a knowledge of X_{n+3} and therefore of x_{n+3} itself. This leads to a corresponding provisional value of Kx_{n+2} by (1) and hence to $x_{n+3}-x_{n+2}$ or x_{n+3}. Then X_{n+3} can be calculated, in general, with the accuracy which is finally necessary. If this be so, KX_{n+2} is now accurately known, and hence x_{n+3} by a simple repetition of the same process, in which if need be an allowance for K^2X can be made. After every few steps in the calculation the whole can be rigorously checked by the difference formula (1) and either verified or corrected if necessary. In general *small* corrections of x_n do not entail a re-adjustment of X_n.

197. This is the principle of the method employed by Cowell and Crommelin in calculating the path of Halley's Comet during the two revolutions 1759–1835–1910. It is the crudest possible method in the sense that it ignores completely what is known of the approximate orbit and is based on the equations of motion in their primitive form, but it is none the less extremely effective for its practical purpose. The origin of coordinates is taken

at the centre of gravity of the solar system, with the axis of x towards the equinox, the axis of y towards longitude 90° and the axis of z towards the N. pole of the ecliptic for a stated fixed epoch. The equations of motion are then (§ 20)

$$m\ddot{x} = -\frac{\partial U}{\partial x}, \quad m\ddot{y} = -\frac{\partial U}{\partial y}, \quad m\ddot{z} = -\frac{\partial U}{dz}$$

where

$$U = -k^2 m \Sigma m_j \{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2\}^{-\frac{1}{2}}$$

and Σ includes the Sun and all the disturbing planets. Thus the typical equation may be written

$$Kx = \left(1 + \frac{1}{12}K - \frac{1}{240}K^2 + \frac{31}{60480}K^3 - \dots\right) X$$

where

$$X = -\Sigma (k^2 w^2 m_j)(x - x_j) r_j^{-3}$$

and $k^2 w^2 m_j$ is a constant for each attracting body. The problem, being in three dimensions, involves the parallel solution of the three similar equations for x, y and z. It is convenient to change the time interval from time to time according to circumstances, in order to economise labour in computing the forces by making the interval as long as experience may show to be practicable. In the example referred to, $w = 2^p$ days, where p has integral values ranging from 1 in the neighbourhood of the Sun to 8 in the most distant part of the orbit. As the comet recedes from the Sun it becomes feasible to treat first Venus and later the Earth and Mars as forming a centrobaric system with the Sun, so that the separate computation of their attractions is avoided. The solution is started by deriving the rectangular coordinates of the comet on two consecutive dates from the osculating elements at the intermediate epoch 1835.

A similar treatment has been applied to the path of Jupiter's eighth satellite, which is so distant from its primary that the solar perturbations are relatively very considerable.

198. The above process is closely related to the more usual method of calculating special perturbations in rectangular coordinates, which dates from Encke. Here the origin is taken at the centre of the Sun and a fixed ecliptic system of axes is generally chosen. Let (x, y, z) be the position of the disturbed body P, (x_j, y_j, z_j) of the typical disturbing planet P_j, and let $SP = r$, $SP_j = \rho_j$ and $PP_j = \Delta_j$. Then the equations of motion of P relative to the Sun are of the form (§ 23)

$$\frac{d^2x}{dt^2} = -k^2(1 + m)\frac{x}{r^3} + k^2 \Sigma m_j \left(\frac{x_j - x}{\Delta_j^3} - \frac{x_j}{\rho_j^3}\right).$$

But the undisturbed motion is given by

$$\frac{d^2x_0}{dt^2} = -k^2(1 + m)\frac{x_0}{r_0^3}$$

where (x_0, y_0, z_0) and r_0 can be calculated at regular intervals of time from the osculating elements. Hence if (ξ, η, ζ) are the perturbations, where

$$\xi = x - x_0, \ldots$$

$$\frac{d^2\xi}{dt^2} = k^2 \left\{\Sigma m_j \left(\frac{x_j - x}{\Delta_j^3} - \frac{x_j}{\rho_j^3}\right) + (1+m)\left(\frac{x_0}{r_0^3} - \frac{x}{r^3}\right)\right\}.$$

The right-hand side contains (ξ, η, ζ) implicitly, and therefore extrapolation is necessary as in § 197. But in the first member ξ, which is of the first order in m_j, is multiplied by m_j and hence if the second order in m_j be neglected (x_0, y_0, z_0) can be directly substituted for (x, y, z). This is consequently known as the direct member, but it is quite possible to include approximate values of the perturbations as they become known in the course of the work, and thus to make allowance for the higher orders of the disturbing masses. The second member, which has been called the indirect member, has no small multiplier and besides is expressed as the difference of two nearly equal quantities. To avoid this inconvenience the transformation

$$\frac{r^2}{r_0^2} = 1 + 2q, \quad \frac{r_0^3}{r^3} = (1+2q)^{-\frac{3}{2}} = 1 - fq$$

is made, where

$$q = (r^2 - r_0^2)/2r_0^2 = \{(x_0 + \tfrac{1}{2}\xi)\,\xi + (y_0 + \tfrac{1}{2}\eta)\,\eta + (z_0 + \tfrac{1}{2}\zeta)\,\zeta\}\, r_0^{-2}$$

$$f = 3\left(1 - \frac{5}{2}q + \frac{5.7}{2.3}q^2 - \frac{5.7.9}{2.3.4}q^3 + \ldots\right) \quad \ldots\ldots\ldots\ldots\ldots(2)$$

and f is tabulated as a function of q, which is a small quantity. The equation for ξ now becomes

$$\frac{d^2\xi}{dt^2} = k^2 \left\{\Sigma m_j \left(\frac{x_j - x}{\Delta_j^3} - \frac{x_j}{\rho_j^3}\right) + \frac{1+m}{r_0^3}(fqx - \xi)\right\}$$

$$= \Sigma X + hfqx - h\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

with parallel equations for η and ζ. This treatment is not applied to the planets with sensible masses, but only to bodies whose masses are negligible and generally unknown. Hence $h = k^2 r_0^{-3}$.

Suppose that $n-1$ steps in the quadrature have been carried out, so that ξ_{n-1}, $\ddot{\xi}_{n-1}$ are known and ξ_n is required. As in § 197 w^2 can be omitted by substituting w^2k^2 for k^2. Then, by (30),

$$\xi_n = \left(K^{-1} + \frac{1}{12} - \frac{1}{240}K + \ldots\right)\ddot{\xi}_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$= \left(K^{-1} - \frac{1}{240}K\right)\ddot{\xi}_n + \frac{1}{12}\Sigma X_n + \frac{1}{12}hfqx_n - \frac{1}{12}h\xi_n$$

or

$$\xi_n\left(1 + \frac{1}{12}h\right) = S_{x,n} + \frac{1}{12}hfqx_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Here $S_{x,n}$ comprises the terms which can be directly calculated, for ΣX_n represents the direct terms, $K^{-1}\ddot{\xi}_n$ follows from the previous stage of the quadrature, and $K\ddot{\xi}_n$ can be extrapolated easily owing to its small multiplier. Also $x_n = x_0 + \xi_n$ is known well enough since it is multiplied by q. But q itself is not accurately known. By combining the three parallel equations of the same type as the last with the above equation for q, it follows that

$$qr_0^2\left(1+\frac{1}{12}h\right)=\Sigma\left(x_0+\frac{1}{2}\xi_n\right)S_{x,n}+\frac{1}{12}hfq\,\Sigma\left(x_0+\frac{1}{2}\xi_n\right)x_n$$

where Σ refers to the three coordinates. Thus, f being easily extrapolated, q can be calculated. The combination of (3) and (5) now gives

$$\ddot{\xi}_n=\Sigma X_n+h\left(1+\frac{1}{12}h\right)^{-1}(fqx_n-S_{x,n})$$

whence $\ddot{\xi}_n$ can be calculated, and therefore ξ_n by (4). Thus the quadrature, once started, proceeds step by step.

In order to start the quadrature the four dates are taken such that the epoch of osculation coincides with the centre of the middle interval. With $\xi=0$ the direct terms in $\ddot{\xi}$ are calculated and the difference table is formed. By applying (27) and (34) approximate values of ξ are obtained whereby the indirect terms can be brought in. The process is then repeated until the final approximation is reached. The rest of the calculation, giving the results by means of (30), has already been explained.

199. Special perturbations may also be directly calculated for polar coordinates. Let the cylindrical coordinates of the disturbed mass m be (ρ, θ, z), the fundamental plane being the plane of the osculating orbit itself at the epoch t_0, and the initial line passing through the ecliptic node. The rectangular coordinates of the typical disturbing planet, of mass m_j, relative to the Sun are

$$x_j=r_j\cos B_j\cos L_j,\quad y_j=r_j\cos B_j\sin L_j,\quad z_j=r_j\sin B_j.$$

The kinetic energy of m is $\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\theta}^2+\dot{z}^2)$, and therefore the equations of motion are, since $r^2=\rho^2+z^2$,

$$\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\theta}{dt}\right)^2=-k^2(1+m)\rho r^{-3}+\frac{\partial R}{\partial\rho},$$

$$\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)=\frac{\partial R}{\partial\theta},\quad \frac{d^2z}{dt^2}=-k^2(1+m)zr^{-3}+\frac{\partial R}{\partial z}$$

where (§ 23)

$$R=k^2\Sigma m_j\{\Delta_j^{-1}-r_j^{-3}(xx_j+yy_j+zz_j)\}$$

$$=k^2\Sigma m_j\{\Delta_j^{-1}-r_j^{-3}[\rho r_j\cos B_j\cos(L_j-\theta)+zr_j\sin B_j]\}$$

$$\Delta_j^2=\rho^2+z^2+r_j^2-2[\rho r_j\cos B_j\cos(L_j-\theta)+zr_j\sin B_j].$$

Hence

$$\ddot{\rho} - \rho\dot{\theta}^2 = -k^2(1+m)\rho r^{-3} - k^2\Sigma m_j\{\rho\Delta_j^{-3} - (\Delta_j^{-3} - r_j^{-3})r_j\cos B_j\cos(L_j-\theta)\}$$

$$d(\rho^2\dot{\theta})/dt = k^2\rho\,\Sigma m_j(\Delta_j^{-3} - r_j^{-3})r_j\cos B_j\sin(L_j-\theta)$$

$$\ddot{z} = -k^2(1+m)zr^{-3} - k^2\Sigma m_j\{z\Delta_j^{-3} - (\Delta_j^{-3} - r_j^{-3})r_j\sin B_j\}.$$

Let

$$z^2/\rho^2 = 2q, \quad \rho^3/r^3 = (1+2q)^{-\frac{3}{2}} = 1 - fq$$

where f is the same function of q as in (2) and can usually be replaced by 3 simply, because z is merely the perturbation in latitude reckoned from the osculating plane. The equations of motion can now be written:

$$\ddot{\rho} - \rho\dot{\theta}^2 + k^2(1+m)\rho^{-2} = \rho H$$

$$d(\rho^2\dot{\theta})/dt = U, \qquad \ddot{z} + W_2 z = W_1$$

where

$$H = \tfrac{1}{2}k^2(1+m)f\rho^{-5}z^2 + k^2\Sigma m_j\{\rho^{-1}(\Delta_j^{-3} - r_j^{-3})r_j\cos B_j\cos(L_j-\theta) - \Delta_j^{-3}\}$$

$$U = k^2\rho\,\Sigma m_j(\Delta_j^{-3} - r_j^{-3})r_j\cos B_j\sin(L_j-\theta)$$

$$W_1 = k^2\Sigma m_j(\Delta_j^{-3} - r_j^{-3})r_j\sin B_j + \tfrac{1}{2}k^2(1+m)f\rho^{-5}z^3$$

$$W_2 = k^2\Sigma m_j\Delta_j^{-3} + k^2(1+m)\rho^{-3}.$$

The third equation is now in the required form to determine z. The first two must be transformed in order to obtain ρ and θ.

200. The second equation gives

$$\rho^2\dot{\theta} = h + \int_{t_0}^{t} U\,dt$$

where h is the undisturbed constant of areas, so that

$$h = \{k^2(1+m)p_0\}^{\frac{1}{2}} = n_0 a_0^2\cos\phi_0$$

p_0, n_0, a_0, $\sin\phi_0$ being the osculating parameter, mean motion, mean distance and eccentricity. Hence

$$\begin{aligned}\theta &= \theta_0 + h\int_{t_0}^{t}\rho^{-2}dt + \int_{t_0}^{t_1}\left[\rho^{-2}\int_{t_0}^{t}U\,dt\right]dt\\ &= \omega_0 + V \qquad\qquad\quad + \Delta\omega\end{aligned}$$

where θ_0 is the initial value of θ and ω_0 is the distance of the undisturbed perihelion from the node. The angle $\Delta\omega$, which represents and is defined by the double integral, would vanish in the absence of disturbing forces. In the same circumstances V would be the undisturbed true anomaly. Thus V may be regarded as the disturbed true anomaly and $\Delta\omega$ as a rotation of the apse.

In the rotating orbit thus defined, in which the elements p_0, a_0, e_0, ϕ_0 keep their osculating values, let $\rho(1+\nu)^{-1}$ be the radius vector corresponding to the true anomaly V, so that, since $\dot{V} = h\rho^{-2}$,

$$1 + e_0\cos V = p_0(1+\nu)\rho^{-1}$$

$$-e_0\sin V = h^{-1}\rho^2 p_0\{-(1+\nu)\rho^{-2}\dot{\rho} + \dot{\nu}\rho^{-1}\}$$

$$-e_0\cos V = h^{-2}\rho^2 p_0\{-(1+\nu)\ddot{\rho} + \rho\ddot{\nu}\}.$$

Hence

$$1 = h^{-2}(1+\nu)\, p_0\rho^2 (h^2\rho^{-3} - \ddot{\rho}) + h^{-2}p_0\rho^3\ddot{\nu}$$

or

$$\ddot{\rho} = h^2\rho^{-3} + (1+\nu)^{-1}\rho\ddot{\nu} - k^2(1+m)(1+\nu)^{-1}\rho^{-2}.$$

But

$$\rho\dot{\theta}^2 = \rho^{-3}\left\{h + \int_{t_0}^{t} U dt\right\}^2.$$

Therefore, by the first equation of motion in the form last found,

$$\rho H = (1+\nu)^{-1}\rho\ddot{\nu} + k^2(1+m)(1+\nu)^{-1}\nu\rho^{-2} - \rho^{-3}\int_{t_0}^{t} U dt \left\{\int_{t_0}^{t} U dt + 2h\right\}$$

which can be written in the form

$$\ddot{\nu} + H_2\nu = H_1$$

where

$$H_1 = H + \rho^{-4}\int_{t_0}^{t} U dt \left\{\int_{t_0}^{t} U dt + 2h\right\}$$

$$H_2 = k^2(1+m)\rho^{-3} - H_1.$$

From this equation, which is of the same form as that in z, ν can be found by mechanical integration.

Again, instead of finding V by a direct quadrature, the necessary correction N is found to the mean anomaly calculated with the undisturbed mean motion n_0, so as to reproduce the true anomaly V or the eccentric anomaly E in the rotating orbit. Thus

$$E - e_0 \sin E = M_0 + n_0(t - t_0) + N$$

$$a_0(1 - e_0\cos E) = \rho(1+\nu)^{-1}.$$

Hence, by (7) of § 27,

$$\dot{N} + n_0 = (1 - e_0\cos E)\dot{E} = \rho a_0^{-1}(1+\nu)^{-1}\dot{V}\,.\,dE/dV$$

$$= \frac{\rho}{a_0(1+\nu)} \cdot \frac{h}{\rho^2} \cdot \frac{1 - e_0\cos E}{\cos\phi_0} = \frac{n_0}{(1+\nu)^2}$$

and

$$\dot{N} = -n_0\nu\,.\,(2+\nu)(1+\nu)^{-2}.$$

201. The whole problem is therefore reduced to the mechanical solution of the equations

$$\frac{d^2\nu}{dt^2} + H_2\nu = H_1, \quad \frac{dN}{dt} = -n_0\nu\,.\,\frac{2+\nu}{(1+\nu)^2}$$

$$\frac{d\Delta\omega}{dt} = \rho^{-2}\int_{t_0}^{t} U dt, \quad \frac{d^2z}{dt^2} + W_2 z = W_1.$$

When ν, N, $\Delta\omega$, z are known, the coordinates r, θ and the latitude λ are given by

$$E - e_0\sin E = M_0 + n_0(t - t_0) + N$$

$$\rho\sin V = (1+\nu)\,a_0\cos\phi_0\sin E, \quad \rho\cos V = (1+\nu)\,a_0(\cos E - e_0)$$

$$\theta = V + \omega_0 + \Delta\omega, \quad r^2 = \rho^2 + z^2, \quad \rho\tan\lambda = z.$$

Perturbations to the first order will be obtained by calculating the quantities occurring in the differential equations according to the osculating elements, but as they become known in the course of the work their approximate effect on the coordinates of the disturbed planet can be introduced before integration. The integral of U, and also N and $\Delta\omega$, can thus be found by direct quadrature by applying (27) and (28). For ν and z, which require exactly similar treatment, the case is slightly different. As before, the time interval w is removed by writing w^2k^2 for k^2, which is equivalent to making this interval the unit of time. Then at any stage n, when z_{n-1} and $K^{-1}\ddot{z}_n$ are known,

$$\ddot{z}_n = W_1 - W_2 z_n$$

$$z_n = \left(K^{-1} + \frac{1}{12} - \frac{1}{240}K + \ldots\right)\ddot{z}_n$$

$$\left(1 + \frac{1}{12}W_2\right)z_n = \left(K^{-1} - \frac{1}{240}K\right)\ddot{z}_n + \frac{1}{12}W_1$$

$$W_2 z_n = W_2\left(1 + \frac{1}{12}W_2\right)^{-1}\left\{\left(K^{-1} - \frac{1}{240}K\right)\ddot{z}_n + \frac{1}{12}W_1\right\}$$

and this last equation will determine $\ddot{z}_n$ with the needful accuracy, and hence z_n and $K^{-1}\ddot{z}_{n+1}$ for the next stage.

This method is due in principle to Hansen. The perturbations start from zero values and remain small for a considerable length of time. This conduces to accuracy and is an advantage. The method is less simple than that of rectangular coordinates, and for the easier construction of an ephemeris requires the determination of new osculating elements by a process which is itself complicated and is omitted here. Perturbations of the coordinates are recommended by the fact that there are three coordinates as against six elements to be determined by quadratures, and their computation is suitable for practical needs in the case of a body, such as a periodic comet, which can only be observed at relatively long intervals. Otherwise it is preferred to perform the calculation on the elements directly.

202. With slight changes which will be readily understood the equations found in § 142 for the perturbations of the elements may be written:

$$di/dt = rW\cos u/k\sqrt{p}$$

$$d\Omega/dt = rW\sin u/k\sqrt{p}\sin i$$

$$d\phi/dt = \{S\sin v + T(\cos v + \cos E)\}\sqrt{p}/k\cos\phi$$

$$d\varpi/dt = \{-pS\cos v + (p+r)T\sin v + rW\sin\phi\tan\tfrac{1}{2}i\sin u\}/k\sqrt{p}\sin\phi$$

$$dn/dt = -3(rS\sin\phi\sin v + pT)\cos\phi/pr$$

$$dM/dt = \{(p\cos v\cos\phi - r\sin 2\phi)S - (p+r)T\sin v\cos\phi\}/k\sqrt{p}\sin\phi + \int_{t_0}^{t}\frac{dn}{dt}dt$$

where v represents the true anomaly and m is neglected, so that $\mu = k^2$. Let

$$wS = kF_1\sqrt{p}, \quad wT = kF_2\sqrt{p}, \quad wW = kF_3\sqrt{p}.$$

Then the equations are of the form

$$wdi/dt = [i, 3]\,F_3, \quad wd\Omega/dt = [\Omega, 3]\,F_3$$
$$wd\phi/dt = [\phi, 1]\,F_1 + [\phi, 2]\,F_2, \quad wdn/dt = [n, 1]\,F_1 + [n, 2]\,F_2$$
$$wd\varpi/dt = [\varpi, 1]\,F_1 + [\varpi, 2]\,F_2 + [\varpi, 3]\,F_3$$
$$w\frac{dM}{dt} = [M, 1]\,F_1 + [M, 2]\,F_2 + w\int_{t_0}^{t}\frac{dn}{dt}\,dt$$

where

$$[i, 3] = r\cos u, \quad [\Omega, 3] = r\sin u/\sin i$$
$$[\phi, 1] = p\sin v\sec\phi, \quad [\phi, 2] = p\,(\cos v + \cos E)\sec\phi$$
$$[\varpi, 1] = -\,p\cos v/\sin\phi, \quad [\varpi, 2] = (p + r)\sin v/\sin\phi, \quad [\varpi, 3] = r\sin u\tan\tfrac{1}{2}i$$
$$[M, 1] = -\,\{[\varpi, 1] + 2r\}\cos\phi, \quad [M, 2] = -\,[\varpi, 2]\cos\phi$$
$$[n, 1] = -\,3k\sin\phi\cos\phi\sin v/\sqrt{p}, \quad [n, 2] = -\,3k\cos\phi\,\sqrt{p}/r.$$

For a minor planet disturbed by Jupiter, 40 days is generally found a suitable value for the interval w.

The disturbing function R may be taken in the form found in § 199 except that the argument of latitude is now $u = v + \varpi - \Omega$ instead of θ. Thus

$$R = k^2\,\Sigma m_j\,\{\Delta_j^{-1} - r_j^{-3}\,[\rho r_j\cos B_j\cos(L_j - u) + zr_j\sin B_j]\}$$

and if the directions of the components S, T, W be recalled,

$$S = \frac{\partial R}{\partial\rho}, \quad T = \frac{1}{\rho}\frac{\partial R}{\partial u}, \quad W = \frac{\partial R}{\partial z}$$

where after differentiation $z = 0$, because the plane of reference is the plane of the instantaneous orbit. For the same reason $\rho = r$. Hence

$$F_1 = p^{-\frac{1}{2}}\,\Sigma\,(kwm_j)\,\{(\Delta_j^{-3} - r_j^{-3})\,r_j\cos B_j\cos(L_j - u) - r\Delta_j^{-3}\}$$
$$F_2 = p^{-\frac{1}{2}}\,\Sigma\,(kwm_j)\,(\Delta_j^{-3} - r_j^{-3})\,r_j\cos B_j\sin(L_j - u)$$
$$F_3 = p^{-\frac{1}{2}}\,\Sigma\,(kwm_j)\,(\Delta_j^{-3} - r_j^{-3})\,r_j\sin B_j$$

and

$$\Delta_j^2 = r^2 + r_j^2 - 2rr_j\cos B_j\cos(L_j - u).$$

203. Let l_j, b_j be the heliocentric longitude and latitude of the disturbing planet, which with log r_j are given in annual tables like the *Nautical Almanac*. The relations between ecliptic coordinates (x, y, z) and the orbital coordinates (ξ, η, ζ), the axis of ξ passing through the ecliptic node, are shown by

	x	y	z
ξ	$\cos\Omega$	$\sin\Omega$	0
η	$-\cos i\sin\Omega$	$\cos i\cos\Omega$	$\sin i$
ζ	$\sin i\sin\Omega$	$-\sin i\cos\Omega$	$\cos i$

which is the scheme derived in § 65. Hence

$$\begin{aligned}\xi &= \cos B_j \cos L_j = &&\cos b_j \cos (l_j - \Omega)\\ \eta &= \cos B_j \sin L_j = &&\cos b_j \cos i \sin (l_j - \Omega) + \sin b_j \sin i\\ \zeta &= \cos B_j \quad\;\; = &&-\cos b_j \sin i \sin (l_j - \Omega) + \sin b_j \cos i\end{aligned}$$

and thus F_1, F_2, F_3 can be calculated, so far as the coordinates of any disturbing planet are concerned.

But F_1, F_2, F_3 and the coefficients $[i, 3]$, ..., involve also the varying elements and coordinates which depend on them. The elements may be identified with the osculating elements at the initial epoch t_0 and the coordinates may be calculated as in undisturbed motion. Then the result of mechanical integration will give the perturbations of the first order. When these are known for the several dates covered by the work, the calculation can be repeated with the improved values and a higher approximation can be obtained. The work can be arranged so as to obviate this repetition by including the perturbations to date at each step.

204. The five elements i, Ω, ϕ, ϖ, n require only a single quadrature. The lower limit $a + \frac{1}{2}w$ is made to coincide with the epoch of osculation and the tables are formed in accordance with (27). The corresponding perturbations are then given by (28) or (26) according as $a + nw$ or $a + (n + \frac{1}{2})\, w$ is preferred for the final date. It is to be noticed that the differential equations for the elements have been reduced to a form in which w occurs explicitly as a coefficient of the derivatives on the left-hand side. It will disappear when the quadratures are effected, its function being to make the unit of time agree with the tabular interval. But the unit of time is not really changed, and with the ordinary Gaussian constant k occurring in the combination kwm_j for each disturbing planet remains one mean solar day. Thus the perturbation in n which will be drawn by this process will be the increment in the mean daily motion. Since all the elements are in the form of angles, it is convenient to express k, so far as it occurs in F_1, F_2, F_3 through the combination kwm_j, by its value in arc ($\log k'' = 3{\cdot}55\ldots$). But in $[n, 1]$, $[n, 2]$ k has its purely numerical value ($\log k = 8{\cdot}235\ldots$).

The perturbation in M can be conveniently divided into two parts. The first,

$$\delta_1 M = w^{-1} \int \{[M, 1]\, F_1 + [M, 2]\, F_2\}\, dt$$

is calculated in precisely the same way as the other five elements. The second is

$$\delta_2 M = \int_{t_0}^{t} \left[\int_{t_0}^{t} \frac{dn}{dt}\, dt\right] dt.$$

The table having been prepared for the first quadrature on the basis of (27) and (28), the second can be performed by means of (34) and (30). The

immediate result will give $w^{-2}\delta_2 M$, which must therefore be multiplied by w^2. To avoid this large multiplier it is usual to calculate $w\delta n$ from $w^2 dn/dt$ at the first quadrature (giving the increment in the w-day mean motion). This alters the time unit of the acceleration and therefore no multiplier will be required by $\delta_2 M$, a result which can be otherwise seen by noticing that all the tabular entries are multiplied by w, while the integrand is divided by w, being in fact dn/dt instead of $w . dn/dt$ as in the first quadrature actually performed on this plan.

205. In the case of parabolic and nearly parabolic orbits some modification is necessary. The equations for i, Ω and ϖ remain valid, except that it is well to replace ϕ by e. The equation for e itself becomes

$$w\,de/dt = [e, 1]\,F_1 + [e, 2]\,F_2$$

$$[e, 1] = p\sin v, \quad [e, 2] = \frac{p}{e}\left(\frac{p}{r} - \frac{r}{a}\right).$$

But the equations for n and M become inconvenient, if not illusory. One suitable substitute is easily obtained by forming the equation for q, the perihelion distance. Since $q = a(1-e)$,

$$w\frac{dq}{dt} = (1-e)\,w\frac{da}{dt} - aw\frac{de}{dt} = -\frac{2aw}{3n}(1-e)\frac{dn}{dt} - aw\frac{de}{dt}$$

$$= [q, 1]\,F_1 + [q, 2]\,F_2$$

where

$$[q, 1] = -\frac{2a}{3n}(1-e)\,[n, 1] - a\,[e, 1]$$

$$= \frac{2ak}{np^{\frac{1}{2}}}\sin\phi\cos\phi\,(1-e)\sin v - ap\sin v$$

$$= 2a^2 e\,(1-e)\sin v - a^2(1-e^2)\sin v = -a^2(1-e)^2\sin v$$

$$= -q^2\sin v$$

and

$$[q, 2] = -\frac{2a}{3n}(1-e)\,[n, 2] - a\,[e, 2]$$

$$= \frac{2ak}{nr}\,p^{\frac{1}{2}}(1-e)\cos\phi - \frac{ap}{e}\left(\frac{p}{r} - \frac{r}{a}\right)$$

$$= \frac{2a^2 p}{r}(1-e) - \frac{ap^2}{er} + \frac{pr}{e}$$

$$= \frac{pr}{e} - \frac{ap^2}{r}\left[\frac{1}{e} - \frac{2}{1+e}\right] = \frac{pr}{e} - \frac{p^3}{r\,.\,e}(1+e)^{-2}$$

$$= \frac{pr}{(1+e)^2}\,.\,4\sin^2\tfrac{1}{2}v\,(1 + e\cos^2\tfrac{1}{2}v).$$

Thus a valid form for the perturbation of q is obtained. If F_1, F_2 have been calculated with the angular value of the constant k the results for δe and δq will require to be multiplied by sin 1″.

Again, an equation can be formed for the variation of T, the time of perihelion passage. Since

$$n(t-T) = M = \epsilon - \varpi + \int n\,dt$$

$$(t-T)\frac{dn}{dt} - n\frac{dT}{dt} = \frac{d}{dt}(\epsilon - \varpi) = \frac{dM}{dt} - \int \frac{dn}{dt}\,dt$$

it follows that

$$w\frac{dT}{dt} = n^{-1}(t-T)\{[n, 1]F_1 + [n, 2]F_2\} - n^{-1}\{[M, 1]F_1 + [M, 2]F_2\}$$
$$= [T, 1]F_1 + [T, 2]F_2$$

where

$$[T, 1] = n^{-1}(t-T)[n, 1] - n^{-1}[M, 1]$$
$$= -\frac{3ke(1-e^2)^{\frac{1}{2}}\sin v\,(t-T)}{np^{\frac{1}{2}}} + \frac{(1-e^2)^{\frac{1}{2}}}{n}\left(2r - \frac{p\cos v}{e}\right)$$
$$= \frac{2(1-e^2)^{\frac{1}{2}}}{n}\left\{r - \frac{p}{2e}\cos v - \frac{3ke}{2p^{\frac{1}{2}}}\sin v\,(t-T)\right\}$$

and

$$[T, 2] = n^{-1}(t-T)[n, 2] - n^{-1}[M, 2]$$
$$= -\frac{3k(1-e^2)^{\frac{1}{2}}p^{\frac{1}{2}}(t-T)}{nr} + \frac{(1-e^2)^{\frac{1}{2}}(p+r)\sin v}{ne}$$
$$= \frac{2(1-e^2)^{\frac{1}{2}}}{n}\left\{\frac{1}{2e}(p+r)\sin v - \frac{3p^{\frac{1}{2}}}{2r}k(t-T)\right\}.$$

But these coefficients are in a form absolutely unsuitable for calculation, especially in the case of a parabola, for which in fact they are required. The difficulty can be, and is best, met for such orbits by calculating special perturbations in rectangular or polar coordinates, instead of directly in the elements.

206. The reduction of $[T, 1]$, $[T, 2]$ to a calculable form is not altogether easy. It can be effected in the following way. The required expressions can be written, since $n^2a^3 = k^2$, $p = a(1-e^2)$,

$$[T, 1] = \frac{2p^{\frac{3}{2}}r}{k(1-e^2)}\left\{1 - \frac{\cos v\,(1+e\cos v)}{2e} - \frac{3k(t-T)}{2p^{\frac{1}{2}}r}e\sin v\right\}$$

$$[T, 2] = \frac{2p^{\frac{3}{2}}r}{k(1-e^2)}\left\{\frac{\sin v\,(2+e\cos v)}{2e} - \frac{3k(t-T)}{2p^{\frac{1}{2}}r}(1+e\cos v)\right\}.$$

Now

$$k(t-T)=a^{\frac{3}{2}}(E-e\sin E)=\frac{p^{\frac{3}{2}}}{(1-e^2)^{\frac{3}{2}}}\{(E-\sin E)+(1-e)\sin E\}$$

$$=\frac{p^{\frac{3}{2}}}{(1+e)^3}\cdot\frac{E-\sin E}{\tan^3\frac{1}{2}E}\cdot\tan^3\tfrac{1}{2}v+\frac{2p^{\frac{3}{2}}}{(1+e)^2}\cos^2\tfrac{1}{2}E\tan\tfrac{1}{2}v$$

$$=\frac{4p^{\frac{3}{2}}}{3(1+e)^3}\cos^2\tfrac{1}{2}E\{(1-S)\tan^2\tfrac{1}{2}v+\tfrac{3}{2}(1+e)\}\tan\tfrac{1}{2}v$$

where

$$1-S=\frac{3(E-\sin E)}{4\tan^3\frac{1}{2}E\cos^2\frac{1}{2}E}=1-\tfrac{1}{20}E^2+\dots.$$

But (§ 27)

$$r\cos^2\tfrac{1}{2}v=a(1-e)\cos^2\tfrac{1}{2}E=p(1+e)^{-1}\cos^2\tfrac{1}{2}E$$

and therefore

$$\frac{3k(t-T)}{2p^{\frac{1}{2}}r}=\frac{\sin v}{(1+e)^2}\{(1-S)\tan^2\tfrac{1}{2}v+\tfrac{3}{2}(1+e)\}=Y.$$

Let $[T,1]$, $[T,2]$ be written in the form

$$[T,1]=\frac{2p^{\frac{3}{2}}r}{k(1-e^2)}\left\{1-\frac{\cos v(1+e\cos v)}{2e}-Y_1\right\}$$

$$[T,2]=\frac{2p^{\frac{3}{2}}r}{k(1-e^2)}\left\{\frac{\sin v(2+e\cos v)}{2e}-Y_2\right\}$$

where

$$Y_1=e\sin v\,.\,Y,\quad Y_2=(1+e\cos v)\,Y$$

and therefore

$$Y_1\cos\tfrac{1}{2}v-Y_2\sin\tfrac{1}{2}v=-(1-e)\sin\tfrac{1}{2}v\,.\,Y$$

$$Y_1\sin\tfrac{1}{2}v+Y_2\cos\tfrac{1}{2}v=\quad(1+e)\cos\tfrac{1}{2}v\,.\,Y.$$

Hence

$$Y_1\cos\tfrac{1}{2}v-Y_2\sin\tfrac{1}{2}v=-\tfrac{1}{2}\sin\tfrac{1}{2}v\sin v\,\frac{1-e}{1+e}\left\{\left(\frac{1-S}{1+e}\cdot 2\tan^2\tfrac{1}{2}v\right)+3\right\}$$

$$Y_1\sin\tfrac{1}{2}v+Y_2\cos\tfrac{1}{2}v=-\tfrac{1}{2}\cos\tfrac{1}{2}v\sin v\left(\frac{S}{1+e}\cdot\frac{2\tan^4\frac{1}{2}v}{\tan^2\frac{1}{2}E}\right)\frac{1-e}{1+e}$$

$$+\tfrac{1}{2}\cos\tfrac{1}{2}v\sin v\{2(1+e)^{-1}\tan^2\tfrac{1}{2}v+3\}.$$

The expressions involving S are finite and they are multiplied by $1-e$, which is a necessary factor. For the other terms, let

$$y_1\cos\tfrac{1}{2}v-y_2\sin\tfrac{1}{2}v=-\tfrac{3}{2}\sin\tfrac{1}{2}v\sin v\,.\,\frac{1-e}{1+e}$$

$$y_1\sin\tfrac{1}{2}v+y_2\cos\tfrac{1}{2}v=\tfrac{3}{2}\cos\tfrac{1}{2}v\sin v+(1+e)^{-1}\cos\tfrac{1}{2}v\sin v\tan^2\tfrac{1}{2}v.$$

Then

$$y_1 = \tfrac{1}{2}(1+e)^{-1}\sin^2 v\,(3e + \tan^2 \tfrac{1}{2}v)$$
$$= \tfrac{1}{2}(1+e)^{-1}(1-\cos v)\{3e(1+\cos v) + (1-\cos v)\}$$
$$y_2 = (1+e)^{-1}\sin v\,\{\tfrac{3}{2}(1+e)\cos^2\tfrac{1}{2}v + \tfrac{3}{2}(1-e)\sin^2\tfrac{1}{2}v + \sin^2\tfrac{1}{2}v\}$$
$$= \tfrac{1}{2}(1+e)^{-1}\sin v\,(4 - \cos v + 3e\cos v).$$

It is now possible to write, with a little simple reduction,

$$[T, 1] = \frac{2p^{\frac{3}{2}} r}{k(1-e^2)}\left\{-\frac{1}{2}\cdot\frac{1-e}{1+e}\left(\cos 2v + \frac{\cos v}{e}\right) + y_1 - Y_1\right\}$$

$$[T, 2] = \frac{2p^{\frac{3}{2}} r}{k(1-e^2)}\left\{\frac{1-e}{1+e}(1+e\cos v)\frac{\sin v}{e} + y_2 - Y_2\right\}$$

and y_1, y_2 have been determined in such a way that

$$(Y_1 - y_1)\cos\tfrac{1}{2}v - (Y_2 - y_2)\sin\tfrac{1}{2}v = -\frac{1}{2}\cdot\frac{1-e}{1+e}\sin v\,.\,g\sin G$$

$$(Y_1 - y_1)\sin\tfrac{1}{2}v + (Y_2 - y_2)\cos\tfrac{1}{2}v = -\frac{1}{2}\cdot\frac{1-e}{1+e}\sin v\,.\,g\cos G$$

where

$$\frac{g\sin G}{\sin\frac{1}{2}v} = \frac{1-S}{1+e}\,.\,2\tan^2\tfrac{1}{2}v,\quad \frac{g\cos G}{\cos\frac{1}{2}v} = \frac{S}{1+e}\cdot\frac{2\tan^4\frac{1}{2}v}{\tan^2\frac{1}{2}E}.$$

Hence

$$[T, 1] = \frac{p^{\frac{3}{2}} r}{k(1+e)^2}\left\{-\cos 2v - \frac{\cos v}{e} + g\sin(G + \tfrac{1}{2}v)\sin v\right\}$$

$$[T, 2] = \frac{p^{\frac{3}{2}}\sin v}{k(1+e)^2}\left\{\frac{2p}{e} + rg\cos(G + \tfrac{1}{2}v)\right\}$$

which are fairly simple forms, but still require the calculation of $g\sin G, g\cos G$. In the limiting case of the parabola, $S = \frac{1}{20}E^2$ and

$$g\sin G = \tan^2\tfrac{1}{2}v\sin\tfrac{1}{2}v,\quad g\cos G = \tfrac{1}{5}\tan^4\tfrac{1}{2}v\cos\tfrac{1}{2}v$$

which then completes the solution.

The more general case of a very eccentric ellipse can be related to the method of § 34. In the notation of that section,

$$A = \frac{15(E - \sin E)}{9E + \sin E},\quad \tau = \tan^2\tfrac{1}{2}E = \frac{A}{1 - \frac{4}{5}A + C}.$$

Hence

$$\frac{10A\sin E}{15 - 9A} = E - \sin E = \tfrac{4}{3}(1-S)\tan^3\tfrac{1}{2}E\cos^2\tfrac{1}{2}E$$

$$1 - S = \frac{15A}{15 - 9A}\cot^2\tfrac{1}{2}E = \frac{1 - \frac{4}{5}A + C}{1 - \frac{3}{5}A}$$

$$S = \frac{\frac{1}{5}A - C}{1 - \frac{3}{5}A},\quad \frac{S}{\tan^2\frac{1}{2}E} = \frac{1 - \frac{4}{5}A + C}{1 - \frac{3}{5}A}\left(\frac{1}{5} - \frac{C}{A}\right).$$

Now by the method of § 34 A (of the order E^2) is found in calculating v, and C (of the order E^4) is tabulated with argument A. With the same argument it is possible to tabulate* log ξ and log η, where

$$1 - S = \xi^{-1}, \quad S \cot^2 \tfrac{1}{2} E = \eta.$$

Then

$$g \sin G = \frac{2 \tan^2 \tfrac{1}{2} v \sin \tfrac{1}{2} v}{(1+e)\,\xi}, \quad g \cos G = \frac{2 \tan^4 \tfrac{1}{2} v \cos \tfrac{1}{2} v}{1+e} \cdot \eta$$

and the problem is thus solved in a practical way. Similar treatment can be applied to hyperbolic orbits.

207. It sometimes happens that a comet approaches a planet (generally Jupiter) so closely that the disturbing force due to the planet is actually greater than the force due to the solar attraction. It is then more convenient to refer the motion to the centre of the planet and to treat the solar action as the disturbing force.

In the ordinary case the equations of motion of the comet are of the form

$$\frac{d^2x}{dt^2} = -k^2M\frac{x}{r^3} + k^2m\left(\frac{x'-x}{\Delta^3} - \frac{x'}{\rho^3}\right)$$

where M is the mass of the Sun, m the mass of the planet, and the origin is at the centre of the Sun. If S, P, C are the positions of Sun, planet and comet, $CS = r$, $CP = \Delta$, $SP = \rho$. The equations involve no assumption as to the relative masses of the Sun and planet, and if they are interchanged the equations of motion of the comet take the form

$$\frac{d^2\xi}{dt^2} = -k^2m\frac{\xi}{\Delta^3} + k^2M\left(\frac{\xi'-\xi}{r^3} - \frac{\xi'}{\rho^3}\right)$$

where the origin is at the centre of the planet, so that $x = x' + \xi, \ldots, x' + \xi' = 0, \ldots$. The advantage of either form depends on the ratio of the total disturbing force to the corresponding central attraction, and it will rest with the latter if

$$\frac{M}{m}\Delta^2\left\{\Sigma\left(\frac{\xi'-\xi}{r^3} - \frac{\xi'}{\rho^3}\right)^2\right\}^{\frac{1}{2}} < \frac{m}{M}r^2\left\{\Sigma\left(\frac{x'-x}{\Delta^3} - \frac{x'}{\rho^3}\right)^2\right\}^{\frac{1}{2}};$$

that is, if $\mu = m/M$, when

$$\Delta^4\left(\frac{1}{r^4} + \frac{1}{\rho^4} - \frac{2}{r^2\rho^2}\cos CSP\right) < \mu^4 r^4\left(\frac{1}{\Delta^4} + \frac{1}{\rho^4} - \frac{2}{\Delta^2\rho^2}\cos CPS\right).$$

Let $CPS = \theta$. Then

$$r \cos CSP = \rho - \Delta \cos\theta$$
$$r^2 = \rho^2 - 2\rho\Delta\cos\theta + \Delta^2.$$

Now in the nature of the case Δ is small compared with ρ. Hence

$$r^{-4} = \rho^{-4} + 4\rho^{-5}\Delta\cos\theta + 2\rho^{-6}\Delta^2(-1 + 6\cos^2\theta) + \ldots$$
$$r^{-3} = \rho^{-3} + 3\rho^{-4}\Delta\cos\theta + \tfrac{3}{2}\rho^{-5}\Delta^2(-1 + 5\cos^2\theta) + \ldots$$

* Bauschinger's *Tafeln*, Nos. XXVII, XXVIII.

and therefore

$$r^{-4} + \rho^{-4} - 2r^{-3}\rho^{-2}(\rho - \Delta\cos\theta) = \rho^{-6}\Delta^2(1 + 3\cos^2\theta) + \ldots.$$

To gain an idea of the planet's *sphere of influence* the approximation need not go further. On the other side of the inequality the first term preponderates and it can be further simplified by taking $r = \rho$. Thus the significant terms of the lowest order in Δ give the inequality

$$\rho^{-6}\Delta^6(1 + 3\cos^2\theta) < \mu^4\rho^4\Delta^{-4}$$

and the polar equation, with coordinates (Δ, θ) and origin at the centre of the planet,

$$\Delta(1 + 3\cos^2\theta)^{\frac{1}{10}} = \mu^{\frac{2}{5}}\rho$$

represents a meridian of the bounding surface, which is one of revolution and differs little from a sphere. Its radius for Jupiter, Saturn and Uranus is about a third, and for Neptune rather more than half, of an astronomical unit.

When the comet enters this sphere of influence its relative coordinates $(x_1 - x_1', y_1 - y_1', z_1 - z_1')$ or (ξ_1, η_1, ζ_1) and its relative velocity $(\dot\xi_1, \dot\eta_1, \dot\zeta_1)$ are known and its orbit about the planet can be found, with the constant of attraction k^2m. It remains within the sphere so short a time that the solar perturbation can generally be neglected, and on emergence a return is made to the heliocentric orbit, based on the new position $(\xi_2 + x_2', \eta_2 + y_2', \zeta_2 + z_2')$ or (x_2, y_2, z_2) and the velocity $(\dot x_2, \dot y_2, \dot z_2)$.

CHAPTER XIX

THE RESTRICTED PROBLEM OF THREE BODIES

208. The general problem of three bodies is reduced to a relatively simple and ideal form when two of the masses describe circles in one plane about their common centre of gravity and the third body has a mass so small as not to affect this circular motion in any appreciable degree. Let $OXYZ$ be a set of rectangular axes rotating with angular velocity n about OZ, OX following OY, and let the coordinates of the masses μ, ν be $(-c_1, 0, 0)$, $(c_2, 0, 0)$ where $\mu c_1 = \nu c_2$. The velocity components in space of a small body at $P(\xi, \eta, \zeta)$ are $(\dot{\xi} - n\eta,\ \dot{\eta} + n\xi,\ \dot{\zeta})$ and hence the kinetic energy of unit mass is

$$T = \tfrac{1}{2}(\dot{\xi} - n\eta)^2 + \tfrac{1}{2}(\dot{\eta} + n\xi)^2 + \tfrac{1}{2}\dot{\zeta}^2.$$

The equations of relative motion are therefore

$$\ddot{\xi} - 2n\dot{\eta} - n^2\xi = \frac{\partial V}{\partial \xi}$$

$$\ddot{\eta} + 2n\dot{\xi} - n^2\eta = \frac{\partial V}{\partial \eta}$$

$$\ddot{\zeta} = \frac{\partial V}{\partial \zeta}$$

where in this case

$$V = k^2(\mu/\rho_1 + \nu/\rho_2)$$

ρ_1, ρ_2 being the distances of P from μ, ν. The result of adding these equations, multiplied respectively by $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, gives Jacobi's integral of energy

$$v^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2V + n^2(\xi^2 + \eta^2) - C$$

and in accordance with Kepler's law

$$k^2(\mu + \nu) = n^2(c_1 + c_2)^3.$$

209. This integral has a very simple and important practical application. Let us return to fixed axes through μ, so that

$$\xi + c_1 = x\cos l + y\sin l, \quad \eta = y\cos l - x\sin l, \quad \zeta = z$$

where l is the longitude of ν and $\dot{l} = n$. Then

$$\dot{\xi}^2 + \dot{\eta}^2 = (\dot{x} + ny)^2 + (\dot{y} - nx)^2$$

$$\xi^2 + \eta^2 = x^2 + y^2 - 2c_1(x\cos l + y\sin l) + c_1^2.$$

Hence Jacobi's integral becomes

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2n(y\dot{x} - x\dot{y}) = 2V - 2n^2c_1(x\cos l + y\sin l) + n^2c_1^2 - C.$$

The special circumstances under which this integral can be usefully employed are these. A periodic comet between two appearances in the neighbourhood of the Sun may pass in close proximity to a large planet, Jupiter for example. In that event the elements may be so altered that at the second return the identity of the comet is doubtful. At times when the perturbations are small and the heliocentric motion is sensibly elliptic,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = k^2(2\mu/\rho_1 - \mu/a)$$
$$x\dot{y} - y\dot{x} = k\sqrt{(\mu p)}\cos i$$

the latter being the projection of the areal velocity on the plane of the disturbing planet. Hence

$$-k^2\mu/a - 2kn\sqrt{(\mu p)}\cos i = 2k^2\nu/\rho_2 - 2n^2c_1(x\cos l + y\sin l) + n^2c_1^2 - C.$$

It is supposed that the change in the observed osculating elements takes place almost impulsively within the region of the planet's influence. This region is small and nearly spherical. Hence ρ_2 is the same at the beginning and end of the encounter, and the changes in x, y and l are small. These can be neglected together with the other planetary perturbations, and therefore approximately

$$\mu/a' + 2k^{-1}n\sqrt{(\mu p')}\cos i' = \mu/a'' + 2k^{-1}n\sqrt{(\mu p'')}\cos i''$$

where a', a'' are the mean distances of the comet, p', p'' the parameters, and i', i'' the inclinations of the orbit to the orbit of the disturbing planet, before and after the encounter. For the Sun $\mu = 1$ and $k^2(1+\nu) = n^2a^3$, where a is the mean distance of the planet, and if ν be neglected

$$a'^{-1} + 2a^{-\frac{3}{2}}p'^{\frac{1}{2}}\cos i' = a''^{-1} + 2a^{-\frac{3}{2}}p''^{\frac{1}{2}}\cos i''$$

which is the criterion of identity proposed by Tisserand. It has been assumed that the orbit of the disturbing planet is circular, but some allowance can be made for the eccentricity of the orbit by taking into account the actual motion of the planet at the time of the suspected encounter.

210. Let the problem of § 208 be now reduced to two dimensions ($\zeta = 0$). Then

$$\mu\rho_1^2 + \nu\rho_2^2 = \mu(\xi + c_1)^2 + \mu\eta^2 + \nu(\xi - c_2)^2 + \nu\eta^2$$
$$= (\mu + \nu)(\xi^2 + \eta^2) + \mu c_1^2 + \nu c_2^2.$$

Let the units be so chosen that $k = 1$ and $c_1 + c_2 = 1$, with the consequence that $\mu + \nu = n^2$. The equations of relative motion may now be written

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial\Omega}{\partial\xi}$$
$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial\Omega}{\partial\eta}$$

where

$$2\Omega = \mu (2\rho_1^{-1} + \rho_1^2) + \nu (2\rho_2^{-1} + \rho_2^2)$$

and the integral of relative energy is

$$v^2 = 2\Omega - C.$$

These are the equations used by Sir G. H. Darwin, with the masses $\mu = 10$, $\nu = 1$, in his researches on periodic orbits. Now it is obvious that v^2 cannot become negative under any circumstances. Hence the curves of the family given in bipolar coordinates by the equation

$$2\Omega = C$$

are of great importance in the restricted problem of three bodies, because they represent barrier curves which cannot be crossed by trajectories characterized by corresponding values of C. Thus if the barrier curve, or curve of zero velocity, is a simple loop within which a part of the trajectory lies, then the trajectory can never pass outside. If the lunar theory can be compared with this simpler problem it is found that the orbit of the Moon lies within such a closed curve surrounding the Earth, and therefore the Moon cannot recede beyond a certain limiting distance from the Earth. This remark is due to Hill.

The simplest view of the general character of the curves of zero velocity is gained by considering them as the contour lines of the surface

$$2\Omega = z, \quad z = C.$$

If the axis of z is taken vertically upwards, and motion for a given value of C is supposed to take place on the actual contour plane $z = C$, then it is evidently restricted to those parts of the plane which lie underneath the surface, since elsewhere in the plane the velocity becomes imaginary. Now the main features of the surface are easily represented topographically. At the points where the masses μ, ν are situated the surface rises to infinity, but in the neighbourhood of these singular points may be treated as two peaks. At any considerable distance from them the terms $\mu\rho_1^2 + \nu\rho_2^2$ are predominant, and the surface rises indefinitely in all directions. Now 2Ω may be expressed in the form

$$2\Omega = 3(\mu + \nu) + \mu(\rho_1 - 1)^2(1 + 2\rho_1^{-1}) + \nu(\rho_2 - 1)^2(1 + 2\rho_2^{-1})$$

and clearly has an absolute minimum value $3(\mu + \nu)$ when $\rho_1 = \rho_2 = 1$, i.e. at the vertices of the equilateral triangle on the line joining the masses μ, ν. These points represent the bottom of two valleys, and a simple consideration of the continuity of the surface shows that these valleys must be connected by three passes, one between the two masses and the others on the same line but on opposite sides of the two masses and separating them from the rising surface as it recedes in the distance. If it be added that the highest pass is

that which lies between the masses and the lowest is on the other side of the greater mass, the general order of development of the contour lines should be sufficiently evident. The critical curves for Darwin's special case, $\mu = 10$, $\nu = 1$, are illustrated in fig. 7. The whole is symmetrical about the line SJ.

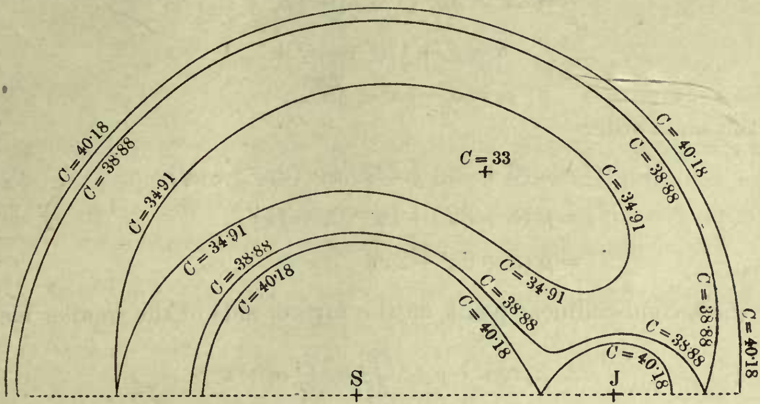

Fig. 7.

211. The points at which the ovals coalesce or disappear evidently correspond to critical values of Ω. Take $\nu < \mu$. The critical values are given by

$$\frac{\partial\Omega}{\partial\xi} = \frac{\partial\Omega}{\partial\rho_1}\cdot\frac{\partial\rho_1}{\partial\xi} + \frac{\partial\Omega}{\partial\rho_2}\cdot\frac{\partial\rho_2}{\partial\xi} = 0$$

$$\frac{\partial\Omega}{\partial\eta} = \frac{\partial\Omega}{\partial\rho_1}\cdot\frac{\partial\rho_1}{\partial\eta} + \frac{\partial\Omega}{\partial\rho_2}\cdot\frac{\partial\rho_2}{\partial\eta} = 0$$

which show immediately that such points are points of relative equilibrium for the third body. These equations are satisfied in the first place by

$$\frac{\partial\Omega}{\partial\rho_1} = \frac{\partial\Omega}{\partial\rho_2} = 0$$

or $\rho_1 = \rho_2 = 1$. This gives the "equilateral" points mentioned above, where Ω is an absolute minimum. But other solutions are given by

$$\frac{\partial(\rho_1, \rho_2)}{\partial(\xi, \eta)} = \frac{1}{\rho_1\rho_2}\begin{vmatrix} \xi + c_1, & \xi - c_2 \\ \eta \;, & \eta \end{vmatrix} = 0$$

or $\eta = 0$, together with $\partial\Omega/\partial\xi = 0$. This will lead to the three points collinear with the masses. For the first, lying between the masses,

$$\rho_1 + \rho_2 = 1, \quad \frac{\partial\rho_1}{\partial\xi} = -\frac{\partial\rho_2}{\partial\xi} = 1$$

so that

$$\frac{\nu}{\mu} = \frac{\rho_1^{-2} - \rho_1}{\rho_2^{-2} - \rho_2} = \frac{\rho_2^2(3\rho_2 - 3\rho_2^2 + \rho_2^3)}{(1 - \rho_2^3)(1 - \rho_2)^2}.$$

This is a quintic in ρ_2, with only one real root. The actual solution in a particular case is easily found by trial and error from the first expression. The second expression, when expanded, gives

$$\frac{\nu}{\mu} = 3\alpha^3 = 3\rho_2^3 (1 + \rho_2 + \tfrac{4}{3}\rho_2^2 + \dots)$$

$$\alpha = \rho_2 + \tfrac{1}{3}\rho_2^2 + \tfrac{1}{3}\rho_2^3 + \dots$$

$$\rho_2 = \alpha - \tfrac{1}{3}\alpha^2 - \tfrac{1}{9}\alpha^3 \dots$$

and to the same order

$$\begin{aligned} C &= \mu (3 + 3\rho_2^2 + 2\rho_2^3) + \nu (2\rho_2^{-1} + \rho_2^2) \\ &= \mu (3 + 3\alpha^2) + \nu\alpha^{-1} (2 + \tfrac{2}{3}\alpha) \\ &= \mu (3 + 9\alpha^2 + 2\alpha^3). \end{aligned}$$

For the second collinear point, on the further side of the smaller mass ν,

$$\rho_1 = 1 + \rho_2, \quad \frac{\partial \rho_1}{\partial \xi} = \frac{\partial \rho_2}{\partial \xi} = +1$$

and hence

$$\frac{\nu}{\mu} = -\frac{\rho_1^{-2} - \rho_1}{\rho_2^{-2} - \rho_2} = \frac{\rho_2^2 (3\rho_2 + 3\rho_2^2 + \rho_2^3)}{(1 - \rho_2^3)(1 + \rho_2)^2}$$

again a quintic in ρ_2 with only one real root. For the approximate solution

$$\frac{\nu}{\mu} = 3\alpha^3 = 3\rho_2^3 (1 - \rho_2 + \tfrac{4}{3}\rho_2^2 - \dots)$$

$$\alpha = \rho_2 - \tfrac{1}{3}\rho_2^2 + \tfrac{1}{3}\rho_2^3 \dots$$

$$\rho_2 = \alpha + \tfrac{1}{3}\alpha^2 - \tfrac{1}{9}\alpha^3 \dots$$

and to the same order

$$\begin{aligned} C &= \mu (3 + 3\rho_2^2 - 2\rho_2^3) + \nu (2\rho_2^{-1} + \rho_2^2) \\ &= \mu (3 + 3\alpha^2) + \nu\alpha^{-1} (2 - \tfrac{2}{3}\alpha) \\ &= \mu (3 + 9\alpha^2 - 2\alpha^3). \end{aligned}$$

For the third collinear point, on the further side of the larger mass μ,

$$\rho_2 = 1 + \rho_1, \quad \frac{\partial \rho_1}{\partial \xi} = \frac{\partial \rho_2}{\partial \xi} = -1$$

and therefore

$$\frac{\nu}{\mu} = -\frac{\rho_1^{-2} - \rho_1}{\rho_2^{-2} - \rho_2} = -\frac{(2 + \sigma)^2 (3\sigma + 3\sigma^2 + \sigma^3)}{(1 + \sigma)^2 (7 + 12\sigma + 6\sigma^2 + \sigma^3)}$$

where $\rho_1 = 1 + \sigma$, $\rho_2 = 2 + \sigma$. Hence

$$\frac{\nu}{\mu} = \frac{-\sigma (12 + 24\sigma + 19\sigma^2 + \dots)}{7 + 26\sigma + 37\sigma^2 + \dots}$$

and

$$\frac{\nu}{\mu + \nu} = \frac{-12\sigma (1 + 2\sigma) - 19\sigma^3 - \dots}{7 (1 + 2\sigma) + 13\sigma^2 + \dots}$$

which shows that

$$\sigma = \frac{-7\nu}{12(\mu+\nu)} = \frac{-7\alpha^3}{4+12\alpha^3}$$

is a very close approximation. The approximate value of C at this point is

$$\begin{aligned} C &= \mu(3+3\sigma^2) + \nu(5+\tfrac{7}{2}\sigma) \\ &= \mu(3+\tfrac{147}{16}\alpha^6) + 3\mu\alpha^3(5-\tfrac{49}{8}\alpha^3) \\ &= \mu(3+15\alpha^3-\tfrac{147}{16}\alpha^6). \end{aligned}$$

When $\nu/\mu = 3\alpha^3$ is small, as in the case of the planets compared with the Sun, the above approximations are generally more than sufficient. In the limiting case $\mu=\nu$ and the arrangement of the points of relative equilibrium is obviously symmetrical with respect to the rotating masses.

212. Let $\xi = \xi_0 + x$, $\eta = \eta_0 + y$, where (ξ_0, η_0) is a fixed point. The equations of motion may then be written

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_{10} + \Omega_{20}x + \Omega_{11}y + \ldots \\ \ddot{y} + 2n\dot{x} &= \Omega_{01} + \Omega_{11}x + \Omega_{02}y + \ldots \end{aligned}$$

where

$$\Omega_{ij} = \frac{\partial^{i+j}\Omega}{\partial\xi_0{}^i\partial\eta_0{}^j}$$

provided Ω is regular at the point (ξ_0, η_0) and x, y are not too large. If (ξ_0, η_0) is a point of relative equilibrium, or as it has been called a point of libration, and x, y are very small, the linear equations

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_{20}x + \Omega_{11}y \\ \ddot{y} + 2n\dot{x} &= \Omega_{11}x + \Omega_{02}y \end{aligned}$$

are obtained, and these determine the nature of the equilibrium at (ξ_0, η_0). For they are satisfied by the solution

$$x = h\cos(mt-\alpha), \quad y = k\cos(mt-\beta)$$

provided

$$\begin{aligned} -2mnk\sin\beta &= (m^2+\Omega_{20})h\cos\alpha + k\Omega_{11}\cos\beta \\ 2mnk\cos\beta &= (m^2+\Omega_{20})h\sin\alpha + k\Omega_{11}\sin\beta \\ 2mnh\sin\alpha &= h\Omega_{11}\cos\alpha + (m^2+\Omega_{02})k\cos\beta \\ -2mnh\cos\alpha &= h\Omega_{11}\sin\alpha + (m^2+\Omega_{02})k\sin\beta. \end{aligned}$$

These equations, which result from equating coefficients of $\cos mt$, $\sin mt$, are equivalent to

$$\begin{aligned} (m^2+\Omega_{20})h\sin(\alpha-\beta) &= 2mnk \\ k\Omega_{11}\sin(\alpha-\beta) &= -2mnk\cos(\alpha-\beta) \\ (m^2+\Omega_{02})k\sin(\alpha-\beta) &= 2mnh \\ h\Omega_{11}\sin(\alpha-\beta) &= -2mnh\cos(\alpha-\beta). \end{aligned}$$

There are only three independent equations here, and this should be so because the only quantities which can be determined are the ratio of

amplitudes h/k, the difference of phases $\alpha - \beta$, and m. The three equations may be written

$$\begin{aligned} h^2(m^2+\Omega_{20}) &= k^2(m^2+\Omega_{02}) \\ \Omega_{11}\tan(\alpha-\beta) &= -2mn \\ (m^2+\Omega_{20})(m^2+\Omega_{02}) &= 4m^2n^2+\Omega^2_{11} \end{aligned}$$

and these determine a series of infinitesimal elliptic orbits about a point of libration when m has a real value. With certain simple developments such a series can be traced into a family of finite periodic orbits.

213. The third equation, that is the quadratic in m^2,

$$m^4 - m^2(4n^2 - \Omega_{20} - \Omega_{02}) + \Omega_{20}\Omega_{02} - \Omega^2_{11} = 0$$

decides the question of stability and may be examined more closely. If the roots in m^2 are complex or negative, real exponential functions of the time enter into the disturbed motion and equilibrium is unstable. If the roots are real, but of opposite sign, an unstable mode of motion is associated with a possible elliptic mode and equilibrium is again unstable. Here the point is surrounded by an unstable family of orbits initially elliptic. This is illustrated by the collinear points of libration. For it is easily found that when $\eta = 0$

$$\Omega_{11} = 0, \quad \Omega_{20} = \mu(2\rho_1^{-3}+1) + \nu(2\rho_2^{-3}+1)$$

so that Ω_{20} is positive. Now at the point of libration between the masses

$$\rho_1 + \rho_2 = 1, \quad \frac{\partial \rho_1}{\partial \xi} + \frac{\partial \rho_2}{\partial \xi} = 1, \quad \frac{\partial \Omega}{\partial \rho_1} = \frac{\partial \Omega}{\partial \rho_2}$$

and therefore, since $\eta = 0$,

$$\Omega_{02} = \frac{1}{\rho_1}\cdot\frac{\partial \Omega}{\partial \rho_1} + \frac{1}{\rho_2}\cdot\frac{\partial \Omega}{\partial \rho_2} = \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\cdot\mu\left(\rho_1 - \frac{1}{\rho_1^2}\right)$$

which is negative since $\rho_1 < 1$. Similarly Ω_{02} is negative at the other collinear points of libration. Hence at these three points the absolute term of the quadratic in m^2 is negative and the roots are real and of opposite sign. Each of the points is therefore surrounded by a family of unstable periodic orbits. It has been suggested by Gyldén and by Moulton that the phenomenon known as the Gegenschein is due to sunlight reflected by meteors which, in spite of the instability, are temporarily retained in the neighbourhood of that centre of libration in the Sun-Earth system which is opposite to the Sun and at a distance of about 938,000 miles from the Earth.

When both values of m^2 are positive the disturbed motion is the resultant of two elliptic motions, and equilibrium is stable. This may be illustrated by the "equilateral" centres of libration. At one of these

$$\frac{\partial \Omega}{\partial \rho_1} = \frac{\partial \Omega}{\partial \rho_2} = \frac{\partial^2 \Omega}{\partial \rho_1 \partial \rho_2} = 0$$

$$\frac{\partial \rho_1}{\partial \xi} = -\frac{\partial \rho_2}{\partial \xi} = \frac{1}{2}, \quad \frac{\partial \rho_1}{\partial \eta} = \frac{\partial \rho_2}{\partial \eta} = \pm\frac{\sqrt{3}}{2}$$

and therefore

$$\Omega_{20} = \left(\frac{\partial \rho_1}{\partial \xi}\right)^2 \frac{\partial^2 \Omega}{\partial \rho_1^2} + \left(\frac{\partial \rho_2}{\partial \xi}\right)^2 \cdot \frac{\partial^2 \Omega}{\partial \rho_2^2} = \tfrac{3}{4}(\mu + \nu)$$

$$\Omega_{02} = \left(\frac{\partial \rho_1}{\partial \eta}\right)^2 \frac{\partial^2 \Omega}{\partial \rho_1^2} + \left(\frac{\partial \rho_2}{\partial \eta}\right)^2 \cdot \frac{\partial^2 \Omega}{\partial \rho_2^2} = \tfrac{9}{4}(\mu + \nu)$$

$$\Omega_{11} = \frac{\partial \rho_1}{\partial \xi} \cdot \frac{\partial \rho_1}{\partial \eta} \cdot \frac{\partial^2 \Omega}{\partial \rho_1^2} + \frac{\partial \rho_2}{\partial \xi} \cdot \frac{\partial \rho_2}{\partial \eta} \cdot \frac{\partial^2 \Omega}{\partial \rho_2^2} = \pm \frac{3\sqrt{3}}{4}(\mu - \nu).$$

Hence the quadratic in m^2 becomes, since $n^2 = \mu + \nu$,

$$m^4 - m^2(\mu + \nu) + \tfrac{27}{4}\mu\nu = 0$$

and the roots are real and positive if

$$(\mu + \nu)^2 > 27\mu\nu$$

an inequality which is satisfied if μ/ν is 25 or greater. In that case the equilateral centres of libration are surrounded by two distinct families of stable periodic orbits which are ellipses in their elementary form, with periods tending to $2\pi/m$. If the masses are more nearly equal, the roots of the equation in m^2 are complex, and no such periodic orbits exist.

Since the masses in the system Sun-Jupiter satisfy the condition of stability, and the disturbing influence of Jupiter predominates over the minor planets, it might be expected that planets would be found in this group approximating to the equilateral configuration. Such planets, with a mean motion nearly equal to that of Jupiter, have actually been discovered.

214. A valuable insight into the general character of the solutions of the problem of three bodies is obtained from the periodic solutions because they repeat themselves after every period. These solutions have therefore been the subject of much laborious study. But such orbits will not be indefinitely permanent unless they are also stable. Hence it is necessary to study them in relation to those orbits which initially differ but little from them.

The original equations of motion give

$$\dot{\xi}\ddot{\eta} - \dot{\eta}\ddot{\xi} + 2n(\dot{\xi}^2 + \dot{\eta}^2) = \dot{\xi}\frac{\partial \Omega}{\partial \eta} - \dot{\eta}\frac{\partial \Omega}{\partial \xi}$$

or

$$\frac{v^3}{R} + 2nv^2 = -v\frac{\partial \Omega}{\partial p} = vN \quad \text{.............................(1)}$$

where R is the radius of curvature of the orbit, δp is an element of the outward drawn normal, and N may be called the component of effective force along the inward normal. Hence if the tangent to the orbit makes the angle ϕ with the axis of ξ, $R = v/\dot{\phi}$ and

$$v(\dot{\phi} + 2n) = -\frac{\partial \Omega}{\partial p}.$$

Also the equation of relative energy gives, when the constant C remains unaltered,

$$v\frac{\partial v}{\partial s}=\dot{v}=\frac{\partial\Omega}{\partial s},\quad \frac{v\partial v}{\partial p}=\frac{\partial\Omega}{\partial p}.$$

Let the undisturbed orbit at P be defined by the quantities s and ϕ, and the corresponding point P' on the neighbouring orbit by δs along the undisturbed orbit and δp normal to it. Then

$$(v+\delta v)^2=\left(v+\frac{d\delta s}{dt}+\dot{\phi}\,\delta p\right)^2+\left(\frac{d\delta p}{dt}-\dot{\phi}\,\delta s\right)^2$$

or to the first order

$$\frac{d\delta s}{dt}+\dot{\phi}\,\delta p=\delta v=\frac{\partial\Omega}{\partial p}\cdot\frac{\delta p}{v}+\frac{\partial\Omega}{\partial s}\cdot\frac{\delta s}{v}$$

$$=-(\dot{\phi}+2n)\,\delta p+v^{-1}\dot{v}\,\delta s.$$

Hence

$$2(\dot{\phi}+n)\,\delta p=v^{-1}\dot{v}\,\delta s-\frac{d\delta s}{dt}=-v\frac{d}{dt}\left(\frac{\delta s}{v}\right)\quad\ldots\ldots\ldots\ldots\ldots(2)$$

Again, let (u, u') be the components of velocity in space of P in directions coinciding with δs, δp. Since these lines are rotating with the absolute velocity $(\dot{\phi}+n)$ the kinetic energy of unit mass at P' is

$$T=\frac{1}{2}\left\{u+\frac{d\delta s}{dt}+(\dot{\phi}+n)\,\delta p\right\}^2+\frac{1}{2}\left\{u'+\frac{d\delta p}{dt}-(\dot{\phi}+n)\,\delta s\right\}^2.$$

Hence Lagrange's equation for δp is

$$\dot{u}'+\frac{d^2\delta p}{dt^2}-2(\dot{\phi}+n)\frac{d\delta s}{dt}-\ddot{\phi}\,\delta s-(\dot{\phi}+n)\,u-(\dot{\phi}+n)^2\,\delta p=\frac{\partial V}{\partial p}+\frac{\partial^2 V}{\partial p^2}\delta p+\frac{\partial^2 V}{\partial p\,\partial s}\delta s.$$

Now this equation must be satisfied when $\delta p=\delta s=0$, and when the terms which do not vanish have been removed, it becomes

$$\frac{d^2\delta p}{dt^2}-2(\dot{\phi}+n)\frac{d\delta s}{dt}-\ddot{\phi}\,\delta s-(\dot{\phi}+n)^2\,\delta p=\frac{\partial^2 V}{\partial p^2}\delta p+\frac{\partial^2 V}{\partial p\,\partial s}\delta s.$$

Also it must be satisfied when $\delta p=0$, $\delta s=v\,\delta t$, where δt is constant, for this also represents a point moving on the unvaried orbit. Thus

$$-2(\dot{\phi}+n)\,\dot{v}-\ddot{\phi}v=\frac{\partial^2 V}{\partial p\,\partial s}\cdot v$$

and therefore

$$\frac{d^2\delta p}{dt^2}-2(\dot{\phi}+n)\left(\frac{d\delta s}{dt}-\frac{\dot{v}}{v}\,\delta s\right)-(\dot{\phi}+n)^2\,\delta p=\frac{\partial^2 V}{\partial p^2}\delta p$$

which owing to (2) becomes

$$\frac{d^2\delta p}{dt^2}+3(\dot{\phi}+n)^2\,\delta p=\frac{\partial^2 V}{\partial p^2}\delta p.$$

But

$$\Omega = V + \tfrac{1}{2}n^2 r^2, \quad \frac{\partial^2 (r^2)}{\partial p^2} = \frac{\partial^2 (r^2)}{\partial \xi^2} = 2.$$

Hence finally

$$\frac{d^2 \delta p}{dt^2} + \Theta \delta p = 0 \quad \text{.................................(3)}$$

where

$$\Theta = n^2 + 3(\dot{\phi} + n)^2 - \frac{\partial^2 \Omega}{\partial p^2}$$

a well-known result due to Hill.

Again, Lagrange's equation for δs is

$$\dot{u} + \frac{d^2 \delta s}{dt^2} + 2(\dot{\phi} + n)\frac{d\delta p}{dt} + \ddot{\phi}\delta p + (\dot{\phi} + n)u' - (\dot{\phi} + n)^2 \delta s = \frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s \partial p}\delta p + \frac{\partial^2 V}{\partial s^2}\delta s$$

which must be satisfied when $\delta p = \delta s = 0$ and also when $\delta p = 0$, $\delta s = v\delta t$. Hence, after removing the terms which are independent of δp and δs and then those which contain δp,

$$\frac{d^2 v}{dt^2} - v(\dot{\phi} + n)^2 = v\frac{\partial^2 V}{\partial s^2} = v\left(\frac{\partial^2 \Omega}{\partial s^2} - n^2\right).$$

This result may be used to give Θ another form, namely

$$\Theta = \frac{1}{v}\frac{d^2 v}{dt^2} + 2n^2 + 2(\dot{\phi} + n)^2 - \nabla^2 \Omega \quad \text{..................(4)}$$

where $\nabla^2 = \partial^2/\partial p^2 + \partial^2/\partial s^2 = \partial^2/\partial \xi^2 + \partial^2/\partial \eta^2$. This form may be more convenient than Hill's because ∇^2 (not to be confounded with the three-dimensional ∇^2) does not depend on any particular direction.

For some purposes it is necessary to take the arc s instead of t as the independent variable. Then (3) becomes

$$v\frac{d}{ds}\left(v\frac{d\delta p}{ds}\right) + \Theta \delta p = 0$$

or again, if $\delta p = v^{-\frac{1}{2}}\delta q$,

$$\frac{d^2 \delta q}{ds^2} + \Psi \delta q = 0$$

where

$$\Psi = v^{-2}\Theta - \tfrac{1}{2}v^{-\frac{1}{2}}\frac{d}{ds}\left(v^{-\frac{1}{2}}\frac{dv}{ds}\right)$$

$$= v^{-2}\Theta - \tfrac{1}{2}v^{-2}\frac{\partial^2 \Omega}{\partial s^2} + \tfrac{3}{4}v^{-4}\left(\frac{\partial \Omega}{\partial s}\right)^2.$$

215. When the unvaried orbit is periodic, Θ is a periodic function of t with the same period T. The equation (3) is therefore a particular case of a linear differential equation with periodic coefficients. Its general theory may be indicated. Since the equation is unaltered when t is replaced by $t + T$, $g(t + T)$ is a solution if $g(t)$ is one. But every solution is a linear combination

of any two others which are independent. Hence if g represents $g(t)$ and G represents $g(t+T)$, g_1, g_2 being any two solutions,

$$G_1 = \alpha g_1 + \beta g_2, \quad G_2 = \gamma g_1 + \delta g_2$$

where α, β, γ, δ are constants, not unrelated. For since g_1, g_2 are two solutions of (3)

$$g_2 \ddot{g}_1 = g_1 \ddot{g}_2$$

and therefore

$$\begin{aligned} g_2 \dot{g}_1 - g_1 \dot{g}_2 &= \text{const.} = G_2 \dot{G}_1 - G_1 \dot{G}_2 \\ &= (g_2 \dot{g}_1 - g_1 \dot{g}_2)(\alpha\delta - \beta\gamma). \end{aligned}$$

Hence $\alpha\delta - \beta\gamma = 1$. Let f_1, f_2 be two other independent solutions. Then

$$g_1 = af_1 + bf_2, \quad g_2 = cf_1 + df_2$$

$$G_1 = aF_1 + bF_2, \quad G_2 = cF_1 + dF_2$$

and the result of eliminating g_1, g_2, G_1, G_2 is

$$F_1 = Af_1 + Bf_2, \quad F_2 = Cf_1 + Df_2$$

where

$$\begin{aligned} (ad - bc) A &= ad\alpha + cd\beta - ab\gamma - bc\delta \\ (ad - bc) B &= bd(\alpha - \delta) + d^2\beta - b^2\gamma \\ (ad - bc) C &= -ac(\alpha - \delta) - c^2\beta + a^2\gamma \\ (ad - bc) D &= -bc\alpha - cd\beta + ab\gamma + ad\delta. \end{aligned}$$

Hence $A + D = \alpha + \delta$ is a constant independent of the choice of particular solutions, as well as $AD - BC = \alpha\delta - \beta\gamma = 1$. But it is now possible to choose b/d and a/c so that $B = C = 0$. Then

$$F_1 = Af_1, \quad F_2 = Df_2, \quad AD = 1$$

and the functions f_1, f_2 are defined by the property that they are multiplied by constants when the argument is increased by the period T. Hence the general solution of the differential equation may be written

$$\delta p = a_1 e^{kt} \phi_1(t) + a_2 e^{-kt} \phi_2(t)$$

where ϕ_1, ϕ_2 are periodic functions with the same period as Θ and $\cosh kT = \frac{1}{2}(\alpha + \delta)$, a constant which can be derived from any pair of independent solutions. The quantities $\pm k$ are what Poincaré has called characteristic exponents. If k is a pure imaginary circular functions are involved and δp has no tendency to increase beyond a certain limit. The periodic orbit is then stable. If on the contrary k is real or complex real exponential functions are involved and δp will increase indefinitely. The orbit is then unstable.

The question of stability therefore involves essentially the determination of k. But this is a matter of great difficulty in general. What is known as Mathieu's equation, generally written in the form

$$\frac{d^2y}{dz^2} + (a + 16q \cos 2z)\, y = 0$$

of which the solutions are elliptic cylinder functions, is only a particular case of the general type (3) and it is the subject of an extensive literature. On the astronomical side the reader may consult Poincaré's *Méthodes Nouvelles*, Tome II. See also Whittaker and Watson, *Modern Analysis*, Ch. XIX.

216. The original equations of motion,

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial \Omega}{\partial \xi}, \quad \ddot{\eta} + 2n\dot{\xi} = \frac{\partial \Omega}{\partial \eta}$$

can also be given a canonical form. Let

$$p_1 = \dot{\xi} - n\eta, \quad p_2 = \dot{\eta} + n\xi$$

$$H = \tfrac{1}{2}(p_1 + n\eta)^2 + \tfrac{1}{2}(p_2 - n\xi)^2 - \Omega + \tfrac{1}{2}C$$

and then evidently

$$\dot{\xi} = \frac{\partial H}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H}{\partial \xi}$$

$$\dot{\eta} = \frac{\partial H}{\partial p_2}, \quad \dot{p}_2 = -\frac{\partial H}{\partial \eta}$$

are equivalent to the above, and they are of the required form. The integral of energy is $H = 0$. Now consider the integral

$$J = \int_{t_0}^{t} (-H + p_1\dot{\xi} + p_2\dot{\eta})\, dt.$$

Between fixed limits its variation will vanish along a trajectory in virtue of the canonical equations. Therefore it is a minimum (or at least stationary) along a trajectory as compared with its value along any neighbouring path. Let the time along any such path be determined by the equation of energy $H = 0$. Then the integral becomes

$$J = \int_{t_0}^{t_1} (p_1\dot{\xi} + p_2\dot{\eta})\, dt$$

$$= \int_{t_0}^{t_1} \{\dot{\xi}^2 + \dot{\eta}^2 + n(\xi\dot{\eta} - \eta\dot{\xi})\}\, dt$$

$$= \int_0^1 \{v\, ds + n(\xi\, d\eta - \eta\, d\xi)\}$$

from which form, since $v^2 = 2\Omega - C$, the time is absent. Now

$$\delta \int v\, ds = \int_0^1 \left(\delta v\, ds + v\frac{d\xi}{ds}\, d\delta\xi + v\frac{d\eta}{ds}\, d\delta\eta\right)$$

$$= \int_0^1 \left\{\delta v\, ds - d\left(v\frac{d\xi}{ds}\right)\delta\xi - d\left(v\frac{d\eta}{ds}\right)\delta\eta\right\}$$

$$+ \left[v\frac{d\xi}{ds}\,\delta\xi + v\frac{d\eta}{ds}\,\delta\eta\right]_0^1$$

and

$$\delta \int_0^1 n\,(\xi d\eta - \eta d\xi) = n \int_0^1 (\delta\xi d\eta - \delta\eta d\xi + \xi d\delta\eta - \eta d\delta\xi)$$

$$= 2n \int_0^1 (\delta\xi d\eta - \delta\eta d\xi) + n\,[\xi\delta\eta - \eta\,\delta\xi]_0^1.$$

Therefore, if $\delta\xi = \delta\eta = 0$ at the limits,

$$\delta J = \int_0^1 \left\{\delta v ds - \delta\xi d\left(v\frac{d\xi}{ds}\right) - \delta\eta d\left(v\frac{d\eta}{ds}\right) + 2n\,(\delta\xi d\eta - \delta\eta d\xi)\right\}.$$

Let the tangent to the orbit make the angle ϕ with the axis of ξ, and let δp be the normal distance to an outer neighbouring curve, so that

$$d\xi = ds \,.\cos\phi, \quad d\eta = ds\,.\sin\phi, \quad \delta\xi = \delta p\,.\sin\phi, \quad \delta\eta = -\,\delta p\,.\cos\phi.$$

Then

$$\delta J = \int_0^1 \{\delta v ds - \sin\phi d\,(v\cos\phi)\,\delta p + \cos\phi d\,(v\sin\phi)\,\delta p + 2n\delta p ds\}$$

$$= \int_0^1 K\delta p ds \quad \text{......(5)}$$

where

$$K = \frac{\partial v}{\partial p} + v\frac{d\phi}{ds} + 2n$$

$$= \frac{1}{v}\frac{\partial\Omega}{\partial p} + \frac{v}{R} + 2n$$

R being the radius of curvature. Along an orbit $K = 0$ therefore, and this is a result already expressed in (1). It is further to be noticed that

$$\frac{\partial K}{\partial p} = \frac{1}{v}\frac{\partial^2\Omega}{\partial p^2} - \left(\frac{1}{v^2}\frac{\partial\Omega}{\partial p} - \frac{1}{R}\right)\frac{\partial v}{\partial p} - \frac{v}{R^2}\frac{\partial R}{\partial p}$$

$$= \frac{1}{v}\left\{\frac{\partial^2\Omega}{\partial p^2} - \left(\frac{1}{v}\frac{\partial\Omega}{\partial p} - \frac{v}{R}\right)\frac{1}{v}\frac{\partial\Omega}{\partial p} - \frac{v^2}{R^2}\right\}$$

$$= \frac{1}{v}\left\{\frac{\partial^2\Omega}{\partial p^2} - \left(\frac{2v}{R} + 2n\right)\left(\frac{v}{R} + 2n\right) - \frac{v^2}{R^2}\right\}$$

when $K = 0$, and since $v = R\dot{\phi}$ comparison with (3) shows that

$$\Theta = -\,v\frac{\partial K}{\partial p}.$$

It follows that the action J round a closed orbit is greater than for any adjacent parallel curve when Θ is positive at every point. In this case the periodic orbit is in general stable. Similarly the action J is a real minimum when Θ is negative at every point. Then, as (3) shows, the periodic orbit is obviously unstable.

217. This remark is due to Prof. Whittaker, who has given another application of equation (5). The quantity K can be calculated for all points on a given curve. Now let K be negative everywhere along a simple closed

curve A. Then by (5) the value of J will be diminished when taken round another curve adjacent to and surrounding A. Again, let the quantity K be positive everywhere along another simple closed curve B external to A. The value of J will also be diminished when taken round a curve adjacent to and surrounded by B. Now consider the aggregate of all the simple closed curves which can be drawn in the ring-shaped space bounded by A and B. There must exist, if the space contains no singularity of Ω, one of these curves which will give a smaller value of J than any other, and it cannot coincide with A or B for any part of its length. It represents therefore a periodic orbit characterized by the constant of energy C, and thus the existence of such an orbit is established when the two curves A and B can be found which satisfy the conditions stated. The orbit is necessarily unstable.

The same author has given another elegant theorem. By Green's theorem

$$\iint \nabla^2 (\log v)\, d\xi d\eta = \int \left[\frac{\partial}{\partial \xi}(\log v)\, d\eta - \frac{\partial}{\partial \eta}(\log v)\, d\xi\right]$$

where the first integral is taken over the area of a closed curve, and the second over its boundary. But if the curve is a trajectory, $K = 0$ and therefore

$$\begin{aligned} 0 &= \frac{\partial}{\partial p}(\log v) + \frac{d\phi}{ds} + \frac{2n}{v} \\ &= \frac{\partial}{\partial \xi}(\log v)\frac{\partial \xi}{\partial p} + \frac{\partial}{\partial \eta}(\log v)\frac{\partial \eta}{\partial p} + \frac{d\phi}{ds} + \frac{2n}{v} \\ &= \frac{\partial}{\partial \xi}(\log v)\frac{d\eta}{ds} - \frac{\partial}{\partial \eta}(\log v)\frac{d\xi}{ds} + \frac{d\phi}{ds} + \frac{2n}{v}. \end{aligned}$$

Hence

$$\begin{aligned} \iint \nabla^2 (\log v)\, d\xi\, d\eta &= -\int\left(\frac{d\phi}{ds} + \frac{2n}{v}\right) ds \\ &= -\int (d\phi + 2n\, dt) \\ &= \phi_0 - \phi_1 + 2n(t_0 - t_1). \end{aligned}$$

This assumes that the enclosed area contains no singularity of the integrand. But this function becomes infinite at the centres of attraction. Surround the mass μ at $(-c_1, 0)$ with a small circle κ_1 of radius ρ. Then since

$$v^2 = 2\Omega - C \sim 2\mu\rho_1^{-1}$$

the integral round the circumference becomes

$$\begin{aligned} \int_{\kappa_1}\left(d\eta\frac{\partial}{\partial \xi} - d\xi\frac{\partial}{\partial \eta}\right)\log v &\sim -\int \frac{1}{4\rho_1^2}\left(d\eta\frac{\partial}{\partial \xi} - d\xi\frac{\partial}{\partial \eta}\right)\rho_1^2 \\ &= -\frac{1}{2\rho^2}\int [(\xi + c_1)\, d\eta - \eta\, d\xi] \\ &= -\pi. \end{aligned}$$

Similarly the corresponding integral round a small circle κ_2 surrounding the mass ν tends to the same limit. Now if the outer boundary contains either of the attracting masses or both, the boundary integral must be diminished by subtracting the integrals taken round κ_1 or κ_2 as the case may be. Hence the final result is

$$\iint\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\log v \,.\, d\xi d\eta = j\pi - \gamma - 2nT$$

where $j = 0$, 1 or 2 according as the loop of the orbit contains neither or one or both of the attracting masses, γ is the total angle through which the tangent to the orbit turns, and T is the time from one end of the loop to the other. In the case of a periodic orbit in the form of a single closed curve $\gamma = 2\pi$.

218. The equations of relative motion are capable of a transformation which is very useful in some cases. This may be deduced from the introduction of conjugate functions in a general form. Let the original equations be

$$\ddot{\xi} - 2n\dot{\eta} - n^2\xi = \frac{\partial V}{\partial \xi}$$

$$\ddot{\eta} + 2n\dot{\xi} - n^2\eta = \frac{\partial V}{\partial \eta}$$

or in the Lagrangian form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) - \frac{\partial T}{\partial \xi} = \frac{\partial V}{\partial \xi}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) - \frac{\partial T}{\partial \eta} = \frac{\partial V}{\partial \eta}$$

where

$$T = \tfrac{1}{2}(\dot{\xi} - n\eta)^2 + \tfrac{1}{2}(\dot{\eta} + n\xi)^2$$

and the integral of energy is

$$\tfrac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) = \tfrac{1}{2}n^2(\xi^2 + \eta^2) + V - h.$$

Now let

$$\xi + \iota\eta = f(u + \iota v), \quad \iota^2 = -1$$

so that

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \quad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}$$

and

$$\frac{d}{dt} = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v}.$$

Also let

$$J = \frac{\partial(\xi, \eta)}{\partial(u, v)} = \frac{\partial \xi}{\partial u}\cdot\frac{\partial \eta}{\partial v} - \frac{\partial \xi}{\partial v}\cdot\frac{\partial \eta}{\partial u}.$$

Then if

$$T = T_2 + T_1 + T_0$$

where the suffix denotes the degree of the terms in $\dot{u}$, $\dot{v}$ (or $\dot{\xi}$, $\dot{\eta}$), it will be found that

$$T_2 = \tfrac{1}{2} J (\dot{u}^2 + \dot{v}^2)$$

$$T_1 = n\dot{u} \left(-\eta \frac{\partial \xi}{\partial u} + \xi \frac{\partial \eta}{\partial u}\right) + n\dot{v} \left(-\eta \frac{\partial \xi}{\partial v} + \xi \frac{\partial \eta}{\partial v}\right)$$

$$T_0 = \tfrac{1}{2} n^2 (\xi^2 + \eta^2).$$

The equations of motion may now be written

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{u}}\right) + \frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{u}}\right) - \frac{\partial T_1}{\partial u} = \frac{\partial T_2}{\partial u} + \frac{\partial T_0}{\partial u} + \frac{\partial V}{\partial u}$$

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{v}}\right) + \frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{v}}\right) - \frac{\partial T_1}{\partial v} = \frac{\partial T_2}{\partial v} + \frac{\partial T_0}{\partial v} + \frac{\partial V}{\partial v}$$

and the integral of energy is

$$T_2 = T_0 + V - h.$$

It can be verified without difficulty that

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{u}}\right) - \frac{\partial T_1}{\partial u} = -2nJ\dot{v}.$$

Also

$$\frac{\partial T_2}{\partial u} + \frac{\partial T_0}{\partial u} + \frac{\partial V}{\partial u} = \frac{1}{2}\frac{\partial J}{\partial u}(\dot{u}^2 + \dot{v}^2) + \frac{\partial T_0}{\partial u} + \frac{\partial V}{\partial u}$$

$$= \frac{1}{J}\frac{\partial J}{\partial u}(T_0 + V - h) + \frac{\partial}{\partial u}(T_0 + V)$$

$$= \frac{1}{J}\cdot\frac{\partial}{\partial u}\{J(T_0 + V - h)\}.$$

Hence the equations of motion become

$$\frac{d}{dt}(J\dot{u}) - 2nJ\dot{v} = \frac{1}{J}\cdot\frac{\partial}{\partial u}\{J(T_0 + V - h)\}$$

$$\frac{d}{dt}(J\dot{v}) + 2nJ\dot{u} = \frac{1}{J}\cdot\frac{\partial}{\partial v}\{J(T_0 + V - h)\}.$$

Now let

$$dt = J\,dT, \quad \Omega' = J\{V + \tfrac{1}{2}n^2(\xi^2 + \eta^2) - h\}$$

and we have

$$\frac{d^2u}{dT^2} - 2nJ\frac{dv}{dT} = \frac{\partial \Omega'}{\partial u}$$

$$\frac{d^2v}{dT^2} + 2nJ\frac{du}{dT} = \frac{\partial \Omega'}{\partial v}$$

with the equation of energy

$$\left(\frac{du}{dT}\right)^2 + \left(\frac{dv}{dT}\right)^2 = 2\Omega'.$$

It is convenient to write

$$f_1 = f(u + \iota v), \quad f_2 = f(u - \iota v), \quad \xi^2 + \eta^2 = f_1 f_2$$

and then

$$J = \left(\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \eta}{\partial u}\right)^2 = \frac{\partial f_1}{\partial u} \cdot \frac{\partial f_2}{\partial u} = f_1' f_2'.$$

219. What is needed when V is the potential due to two masses μ, ν at a distance $2c$ apart is a transformation of the coordinates which will rationalize both the distances ρ_1, ρ_2. Such a transformation is

$$\xi + i\eta = b + c \cos(u + iv), \quad b = c(\mu - \nu)/(\mu + \nu)$$

where b is the distance of the middle point between the masses from their centre of gravity. For

$$\rho_1^2 = (\xi - b + c)^2 + \eta^2 = 4c^2 \cos^2 \tfrac{1}{2}(u + iv) \cos^2 \tfrac{1}{2}(u - iv)$$

$$\rho_2^2 = (\xi - b - c)^2 + \eta^2 = 4c^2 \sin^2 \tfrac{1}{2}(u + iv) \sin^2 \tfrac{1}{2}(u - iv)$$

and hence

$$V = \frac{\mu}{\rho_1} + \frac{\nu}{\rho_2} = \frac{\mu}{c(\cosh v + \cos u)} + \frac{\nu}{c(\cosh v - \cos u)}.$$

Also

$$J = f_1' f_2' = c^2 \sin(u + iv) \sin(u - iv) = \tfrac{1}{2}c^2(\cosh 2v - \cos 2u)$$

and

$$\xi^2 + \eta^2 = f_1 f_2 = b^2 + 2bc \cosh v \cos u + \tfrac{1}{2}c^2(\cosh 2v + \cos 2u).$$

Hence

$$\begin{aligned}\Omega' = {} & \mu c(\cosh v - \cos u) + \nu c(\cosh v + \cos u) \\ & + \tfrac{1}{4}n^2 bc^3(\cosh 3v \cos u - \cosh v \cos 3u) + \tfrac{1}{16}n^2 c^4(\cosh 4v - \cos 4u) \\ & - \tfrac{1}{2}c^2(h - \tfrac{1}{2}n^2 b^2)(\cosh 2v - \cos 2u)\end{aligned}$$

and the equations of motion are

$$\frac{d^2u}{dT^2} - nc^2(\cosh 2v - \cos 2u)\frac{dv}{dT} = \frac{\partial \Omega'}{\partial u}$$

$$\frac{d^2v}{dT^2} + nc^2(\cosh 2v - \cos 2u)\frac{du}{dT} = \frac{\partial \Omega'}{\partial v}.$$

The time is given by a final integration

$$t = \tfrac{1}{2}c^2 \int (\cosh 2v - \cos 2u)\, dT = \int \rho_1 \rho_2 \, dT.$$

These equations are in general very complicated, although they offer essential advantages in studying the motion in the immediate vicinity of

one of the masses. Two particular cases may be noticed. In the first the masses are equal, $\mu = \nu$ and $b = 0$. The equations of motion then become

$$\frac{d^2u}{dT^2} - nc^2 (\cosh 2v - \cos 2u) \frac{dv}{dT} = -c^2h \sin 2u + \tfrac{1}{4} n^2c^4 \sin 4u$$

$$\frac{d^2v}{dT^2} + nc^2 (\cosh 2v - \cos 2u) \frac{du}{dT} = 2\mu c \sinh v - c^2h \sinh 2v + \tfrac{1}{4} n^2c^4 \sinh 4v$$

which are equivalent to equations given by Thiele and employed by Strömgren and Burrau. The other case represents the problem of two centres of attraction fixed in space, so that $n = 0$. Then the equations become simply

$$\frac{d^2u}{dT^2} = (\mu - \nu) c \sin u - c^2h \sin 2u$$

$$\frac{d^2v}{dT^2} = (\mu + \nu) c \sinh v - c^2h \sinh 2v.$$

Here the variables u, v are separated and the equations lead immediately to a solution in elliptic functions. The comparison of this problem with the simplest case of the problem of three bodies is instructive as to the difficulty of the latter.

CHAPTER XX

LUNAR THEORY I

220. The theory of the Moon's motion relative to the Earth has been discussed with generally increasing elaboration and completeness by various authors from the time of Newton to the present day. The methods which have been employed also differ considerably, presenting peculiar advantages in different respects, so that it cannot be said definitely that any one method possesses an exclusive claim to consideration. But at the present time three modes of treatment are certainly of outstanding importance, those adopted by Hansen, Delaunay and G. W. Hill respectively. Hansen's theory was reduced to the form of tables by the author; these tables were published in 1857 and are still in common use, but will shortly be superseded. Delaunay's work took the form of an entirely algebraic development of the Moon's motion as conditioned by the Earth and Sun alone. His theory has been completed by others and made the basis of tables recently published. Hill's researches, which bear a certain relation to Euler's memoir of 1772, deal only with particular parts of the theory, but the whole work on these lines has now been carried out systematically and completely by E. W. Brown and will form the foundation of a new set of lunar tables now in course of preparation.

Here it is only possible to attempt a slight sketch of one method. For this purpose Hill's theory will be chosen, partly because it is destined to receive extensive practical application, and partly because it contains original features of the greatest theoretical interest. The reader who wishes to gain a comparative view of the different methods which have been used in the lunar theory will study Brown's *Lunar Theory* and may also be referred to the third volume of Tisserand's *Mécanique Céleste*.

221. Let the mass of the Earth be E, of the Moon M and of the Sun m', the unit being such that the gravitational constant $G = 1$. Let the origin of rectangular axes be E, (x, y, z) the coordinates of M and (x', y', z') the coordinates of m'. Further, let r be the distance EM, r' the distance Em', and Δ the distance Mm'. Then (§ 23) the forces on the Moon per unit mass relative to E can be derived from the force function

$$F = \frac{E+M}{r} + \frac{m'}{\Delta} - \frac{m'}{r'^3}(xx' + yy' + zz')$$

by differentiation with respect to x, y, z; and similarly the forces on the Sun per unit mass relative to E can be derived from the function

$$F' = \frac{E+m'}{r'} + \frac{M}{\Delta} - \frac{M}{r^3}(xx' + yy' + zz')$$

by differentiation with respect to x', y', z'. Hence the x-component of the Sun's acceleration relative to G, the centre of gravity of E and M, is

$$\begin{aligned}\frac{\partial F'}{\partial x'} - \frac{M}{E+M}\frac{\partial F}{\partial x} &= -(E+m')\frac{x'}{r'^3} - M\frac{x'-x}{\Delta^3} - M\frac{x}{r^3} \\ &\quad + \frac{M}{E+M}\left\{(E+M)\frac{x}{r^3} + m'\frac{x-x'}{\Delta^3} + m'\frac{x'}{r'^3}\right\} \\ &= -\frac{E+M+m'}{E+M}\left\{E\frac{x'}{r'^3} + M\frac{x'-x}{\Delta^3}\right\}.\end{aligned}$$

This expression will be derived by differentiating the function

$$F_1' = \frac{E+M+m'}{E+M}\left(\frac{E}{r'} + \frac{M}{\Delta}\right)$$

with respect to x', or with respect to x_1, where (x_1, y_1, z_1) are the new coordinates of m' when parallel axes are taken through G instead of E. Let r_1 be the distance $m'G$, θ_1 the angle $m'GM$ and $S = \cos\theta_1$. Then

$$\begin{aligned}r'^{-1} &= \left\{r_1^2 + \frac{2M}{E+M}rr_1S + \frac{M^2}{(E+M)^2}r^2\right\}^{-\frac{1}{2}} \\ &= r_1^{-1}\,1 - \left\{\frac{M}{E+M}\frac{r}{r_1}P_1 + \frac{M^2}{(E+M)^2}\frac{r^2}{r_1^2}P_2 - \dots\right\}\end{aligned}$$

and

$$\begin{aligned}\Delta^{-1} &= \left\{r_1^2 - \frac{2E}{E+M}rr_1S + \frac{E^2}{(E+M)^2}r^2\right\}^{-\frac{1}{2}} \\ &= r_1^{-1}\left\{1 + \frac{E}{E+M}\frac{r}{r_1}P_1 + \frac{E^2}{(E+M)^2}\frac{r^2}{r_1^2}P_2 + \dots\right\}\end{aligned}$$

where $P_1, P_2, \dots$ are Legendre's polynomials

$$P_1 = S, \quad P_2 = \tfrac{3}{2}S^2 - \tfrac{1}{2}, \quad P_3 = \tfrac{5}{2}S^3 - \tfrac{3}{2}S, \dots.$$

Hence, when expanded in terms of r/r_1,

$$F_1' = \frac{E+M+m'}{r_1}\left\{1 + \frac{EM}{(E+M)^2}\frac{r^2}{r_1^2}P_2 + \dots\right\}.$$

Now the Moon's parallax is of the order $1°$, the solar parallax is of the order $9''$ and the ratio M/E is of the order $1/80$. It follows that the second term in F_1' is of the order 10^{-7} as compared with the first. It can be neglected, at least in the first instance. F_1' is therefore reduced simply to the first term, and the meaning of this is that the motion of G about m', or of m' about G, is the same as if the masses E and M were united at their centre of gravity.

This motion is elliptic and the coordinates (x_1, y_1, z_1) can be treated as known functions of the time according to undisturbed elliptic motion. The influence of the other planets is left out of account in the first instance and finally introduced in the form of small corrections. The first task, and the only one considered here, is to find an appropriate solution of the problem of three bodies, the problem being already so far simplified that the relative motion of the Sun and the centre of gravity of the Earth-Moon system is supposed known.

222. The force function F is expressed in terms of (x', y', z') and not the coordinates (x_1, y_1, z_1) now supposed known. It is necessary to consider the effect of this. The x-component of the Moon's acceleration is

$$\begin{aligned}\frac{\partial F}{\partial x} &= -(E+M)\frac{x}{r^3} - m'\frac{x-x'}{\Delta^3} - m'\frac{x'}{r'^3}\\ &= -(E+M)\frac{x}{r^3} - \frac{m'}{\Delta^3}\left(\frac{E}{E+M}x - x_1\right) - \frac{m'}{r'^3}\left(\frac{M}{E+M}x + x_1\right)\end{aligned}$$

since

$$x' = x_1 + Mx/(E+M), \quad x - x' = -x_1 + Ex/(E+M).$$

This component is clearly derivable from the force function

$$F_1 = \frac{E+M}{r} + \frac{m'(E+M)}{E\Delta} + \frac{m'(E+M)}{Mr'}$$

when r' and Δ are expressed in terms of (x_1, y_1, z_1) instead of (x', y', z'). When Δ^{-1}, r'^{-1} are expanded in terms of r/r_1 this becomes

$$\begin{aligned}F_1 &= \frac{E+M}{r} + \frac{m'}{r_1}\left\{\frac{(E+M)^2}{EM} + \frac{r^2}{r_1^2}P_2 + \frac{E^2-M^2}{(E+M)^2}\frac{r^3}{r_1^3}P_3 + \frac{E^3+M^3}{(E+M)^3}\frac{r^4}{r_1^4}P_4 + \dots\right\}\\ &= \frac{E+M}{r} + \frac{m'r^2}{r_1^3}\left\{P_2 + \frac{E-M}{E+M}\frac{r}{r_1}P_3 + \frac{E^2-EM+M^2}{(E+M)^2}\frac{r^2}{r_1^2}P_4 + \dots\right\}\end{aligned}$$

for the term in $1/r_1$ does not contain (x, y, z) and can therefore be suppressed.

As a matter of fact the force function which is commonly used for the motion of the Moon is neither F_1 nor the function

$$F = \frac{E+M}{r} + \frac{m'}{\Delta} - \frac{m'r}{r'^2}\cos\theta$$

where θ is the angle $m'EM$, but the function

$$F_2 = \frac{E+M}{r} + \frac{m'}{\Delta_1} - \frac{m'r}{r_1^2}S$$

which is derived from F by substituting the coordinates of the Sun relative to G for the coordinates relative to E. Thus

$$\begin{aligned}\Delta_1^2 &= (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\\ &= r^2 - 2rr_1S + r_1^2\end{aligned}$$

and therefore in the expanded form

$$F_2 = \frac{E+M}{r} + \frac{m'}{r_1}\left\{1 + \frac{r}{r_1}P_1 + \frac{r^2}{r_1^2}P_2 + \dots\right\} - \frac{m'r}{r_1^2}S$$

$$= \frac{E+M}{r} + \frac{m'r^2}{r_1^3}\left\{P_2 + \frac{r}{r_1}P_3 + \frac{r^2}{r_1^2}P_4 + \dots\right\}$$

after suppressing m'/r_1. This is not the same as F_1, but for practical purposes it can be brought into agreement by a simple device. Let a, a' be the mean values of r, r_1. It is found that to a term of the series involving $(r/r_1)^j$ correspond inequalities with the factor $(a/a')^j$. If then

$$(E-M)\,a/(E+M)\,a'$$

be substituted for a/a' in the results which follow from the use of F_2, they will be very nearly the same as if they had been derived by using F_1. It may be left to the reader to examine the order of the chief outstanding discrepancy after this treatment of F_2. It is easy to make the adjustment exact.

223. Let the axis Ez be taken normal to the ecliptic and let EX, EY rotate in the ecliptic plane of (xy) with the Sun's mean motion n'. The equations of motion of the Moon are then

$$\ddot{X} - 2n'\dot{Y} - n'^2X = \frac{\partial F_2}{\partial X}$$

$$\ddot{Y} + 2n'\dot{X} - n'^2Y = \frac{\partial F_2}{\partial Y}$$

$$\ddot{z} = \frac{\partial F_2}{\partial z}.$$

Now if $E + M = \mu$, since $n'^2a'^3 = m'$ (more strictly $m' + \mu$),

$$F_2 = \frac{\mu}{r} + n'^2\frac{a'^3}{r_1^3}\left(\tfrac{3}{2}r^2S^2 - \tfrac{1}{2}r^2\right) + \dots$$

the higher terms containing r/r_1 and therefore the solar parallax as a factor. Let v' be the true longitude of the Sun and let $v' = \epsilon'$ when $t = 0$. Then the Sun's coordinates are

$$X' = r_1\cos(v' - n't - \epsilon'),\quad Y' = r_1\sin(v' - n't - \epsilon'),\quad z' = 0$$

the axis of X being always directed towards the Sun's mean place. When the solar eccentricity is neglected and the Sun's orbit treated as circular, $v' = n't + \epsilon'$ and $r_1 = a'$, so that

$$X' = r_1 = a',\quad Y' = z' = 0,\quad rS = (XX' + YY')/r_1 = X.$$

Hence when the solar parallax and eccentricity are both neglected

$$F_2 = \mu r^{-1} + n'^2\left(\tfrac{3}{2}X^2 - \tfrac{1}{2}r^2\right) = \mu r^{-1} + n'^2\left(X^2 - \tfrac{1}{2}Y^2 - \tfrac{1}{2}z^2\right)$$

and when, still further, the latitude of the Moon is ignored, the equations of motion become simply

$$\left.\begin{aligned}\ddot{X} - 2n'\dot{Y} - 3n'^2X &= -\mu X/r^3\\ \ddot{Y} + 2n'\dot{X} \qquad\qquad &= -\mu Y/r^3\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

These two-dimensional equations represent the simplest problem bearing any real resemblance to the actual circumstances of the lunar theory. It is the degenerate case of the restricted problem of three bodies when the two finite masses are relatively at a very great distance apart and refers strictly to the motion of a satellite in the immediate neighbourhood of its primary. These equations have great importance in Hill's theory.

Again, when the solar parallax alone is neglected, F_2 may be written in the form

$$F_2 = \mu r^{-1} + n'^2(\tfrac{3}{2}X^2 - \tfrac{1}{2}r^2) + n'^2\left\{\tfrac{3}{2}\left(\frac{a'^3}{r_1^3}r^2S^2 - X^2\right) - \tfrac{1}{2}r^2\left(\frac{a'^3}{r_1^3} - 1\right)\right\}$$

where the third term, which vanishes with the solar eccentricity, is a quadratic function in X, Y, z. Thus

$$F_2 = \mu r^{-1} + n'^2(X^2 - \tfrac{1}{2}Y^2 - \tfrac{1}{2}z^2) - \tfrac{1}{2}(A'X^2 + 2H'XY + B'Y^2 + C'z^2)$$

where A', H', B', C' are functions of t to be derived from the elliptic motion of the Sun. The equations of motion now become

$$\begin{aligned}\ddot{X} - 2n'\dot{Y} - 3n'^2X + A'X + H'Y &= -\mu X/r^3\\ \ddot{Y} + 2n'\dot{X} \qquad\quad + H'X + B'Y &= -\mu Y/r^3\\ \ddot{z} \qquad\quad + n'^2z \quad + C'z \qquad\quad &= -\mu z/r^3\end{aligned}$$

and these are the foundation of the researches of Adams into the principal part of the motion of the lunar node.

224. It is now necessary to give Hill's transformation of the general equations of motion. Let

$$u = X + \iota Y, \quad s = X - \iota Y, \quad \iota^2 = -1$$

$$\mathrm{m} = \frac{n'}{n - n'}, \quad \kappa = \frac{\mu}{(n - n')^2}, \quad \nu = n - n'.$$

Then, since $r^2 = us + z^2$, n being undefined as yet,

$$\begin{aligned}2\nu^{-2}F_2 &= 2\kappa/r + 2\mathrm{m}^2\frac{a'^3}{r_1^3}(P_2r^2 + P_3r^3/r_1 + \ldots)\\ &= 2\kappa/r + \Omega_2' + \Omega_3 + \ldots\end{aligned}$$

where Ω_2', Ω_3, ... are homogeneous functions in u, s, z of degree 2, 3, ... and of degree 0, -1, ... in a'. Let $\Omega' = \Omega_2' + \Omega_3 + \ldots$.

The kinetic energy of the Moon T is given by

$$\begin{aligned}2T/M &= (\dot{X} - n'Y)^2 + (\dot{Y} + n'X)^2 + \dot{z}^2\\ &= (\dot{u} + n'\iota u)(\dot{s} - n'\iota s) + \dot{z}^2.\end{aligned}$$

The equations of motion are therefore

$$\ddot{u} + 2n'\iota\dot{u} - n'^2 u = 2\frac{\partial F_2}{\partial s}$$

$$\ddot{s} - 2n'\iota\dot{s} - n'^2 s = 2\frac{\partial F_2}{\partial u}$$

$$\ddot{z} = \frac{\partial F_2}{\partial z}.$$

Let

$$\log\zeta = \iota(n-n')(t-t_0), \quad D = \zeta\frac{d}{d\zeta} = -\frac{\iota}{\nu}\frac{d}{dt}$$

where t_0, like n, is a constant at present undefined. The previous equations become

$$D^2u + 2\mathrm{m}Du + \mathrm{m}^2u = \kappa u/r^3 - \frac{\partial\Omega'}{\partial s}$$

$$D^2s - 2\mathrm{m}Ds + \mathrm{m}^2s = \kappa s/r^3 - \frac{\partial\Omega'}{\partial u}$$

$$D^2z = \kappa z/r^3 - \tfrac{1}{2}\frac{\partial\Omega'}{\partial z}.$$

It is, however, convenient to separate from Ω_2' (accented for this reason) the part which is independent of the solar eccentricity. This is

$$\Omega_2' - \Omega_2 = \mathrm{m}^2(3X^2 - r^2) = \tfrac{3}{4}\mathrm{m}^2(u+s)^2 - \mathrm{m}^2(us+z^2).$$

With this change the equations of motion take the form

$$\left.\begin{aligned} D^2u + 2\mathrm{m}Du + \tfrac{3}{2}\mathrm{m}^2(u+s) - \frac{\kappa u}{r^3} &= -\frac{\partial\Omega}{\partial s} \\ D^2s - 2\mathrm{m}Ds + \tfrac{3}{2}\mathrm{m}^2(u+s) - \frac{\kappa s}{r^3} &= -\frac{\partial\Omega}{\partial u} \\ D^2z - \mathrm{m}^2z - \frac{\kappa z}{r^3} &= -\tfrac{1}{2}\frac{\partial\Omega}{\partial z} \end{aligned}\right\} \quad \text{............(2)}$$

where $\Omega = \Omega_2 + \Omega_3 + \ldots$. Thus

$$\Omega_2 = 3\mathrm{m}^2\left\{\frac{a'^3}{r_1^3}r^2S^2 - \tfrac{1}{4}(u+s)^2\right\} - \mathrm{m}^2r^2\left(\frac{a'^3}{r_1^3} - 1\right) \quad \text{............(3)}$$

which vanishes with the solar eccentricity.

225. The next object is to transform the equations in u and s so as to remove the terms involving r^{-3}. Since (§ 123)

$$\frac{d}{dt}(T_2 - T_0 + U) = \frac{\partial U}{\partial t}$$

and F_2 contains terms involving t explicitly only in Ω, in this case

$$\dot{u}\dot{s} + \dot{z}^2 - n'^2us = 2F_2 - \nu^2\int\frac{\partial\Omega}{\partial t}dt + h$$

or in the later notation

$$Du\,.\,Ds + (Dz)^2 + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 - \mathrm{m}^2z^2 + \frac{2\kappa}{r} = C - \Omega + D^{-1}(D_t\Omega)$$

where C is a constant of integration, D^{-1} is the inverse operator to D, and D_t represents the operator D applying to Ω only in so far as Ω contains t explicitly. This corresponds to the equation of energy.

Again, since $r^2 = us + z^2$, the equations of motion (2) give

$$sD^2u + uD^2s + 2zD^2z + 2\mathrm{m}(sDu - uDs) + \tfrac{3}{2}\mathrm{m}^2(u+s)^2 - 2\mathrm{m}^2z^2 - 2\kappa/r$$
$$= -\left(s\frac{\partial\Omega}{\partial s} + u\frac{\partial\Omega}{\partial u} + z\frac{\partial\Omega}{\partial z}\right) = -\sum_{p=2} p\Omega_p$$

by Euler's theorem, Ω_p being a homogeneous function of degree p in u, s, z. The result of adding the last two equations is

$$D^2(us+z^2) - Du\,.\,Ds - (Dz)^2 + 2\mathrm{m}(sDu - uDs) + \tfrac{3}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2z^2$$
$$= C - \sum_{p=2}(p+1)\,\Omega_p + D^{-1}(D_t\Omega) \quad \text{.........(4)}$$

This is one equation of the required form.

The other equations are obtained simply by eliminating the terms with r^{-3} as a factor between different pairs of the equations of motion. Thus from the first pair

$$D(uDs - sDu - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = s\frac{\partial\Omega}{\partial s} - u\frac{\partial\Omega}{\partial u} \quad \text{.........(5)}$$

and when the third equation is used,

$$D(uDz - zDu) - 2\mathrm{m}zDu - \tfrac{1}{2}\mathrm{m}^2z(5u+3s) = z\frac{\partial\Omega}{\partial s} - \tfrac{1}{2}u\frac{\partial\Omega}{\partial z}$$

$$D(sDz - zDs) + 2\mathrm{m}zDs - \tfrac{1}{2}\mathrm{m}^2z(3u+5s) = z\frac{\partial\Omega}{\partial u} - \tfrac{1}{2}s\frac{\partial\Omega}{\partial z}.$$

These combined give

$$D\{(u\pm s)Dz - zD(u\pm s)\} - 2\mathrm{m}zD(u\mp s) - \mathrm{m}^2zW$$
$$= z\left(\frac{\partial\Omega}{\partial s} \pm \frac{\partial\Omega}{\partial u}\right) - \tfrac{1}{2}(u\pm s)\frac{\partial\Omega}{\partial z}$$

where with the upper sign $W = 4(u+s)$ and with the lower $W = u - s$. In this more symmetrical form the real and imaginary parts of u and s are clearly separated.

Equations in the form of (4) and (5) have two advantages. In the first place the left-hand members are homogeneous in u, s, z of the second degree. Except for the constant C this applies also to the right-hand members when the parallax of the Sun is neglected, and the parallactic terms need rarely be taken beyond the third and fourth degrees. In the second place, whereas X and Y can be expressed as trigonometrical series in terms of t, u and s can be expressed as algebraic (Laurent) series in terms of ζ, and such series

can be more easily manipulated. Also if $u = f(\zeta)$, $s = f(\zeta^{-1})$ and therefore when either u or s has been calculated the other can be derived immediately.

226. The general method of the lunar theory, which is common to all forms, consists in choosing an intermediate orbit which bears some resemblance to the actual path of the Moon and in studying the variations which it must undergo in order that the path may be represented accurately and permanently. This intermediate orbit, since it merely serves as a subject for amendment, will naturally be chosen with a view to simplicity. At the same time, the more closely it represents the permanent features of the actual motion, the less burden will be thrown on the subsequent variations. Thus one might take the osculating elliptic orbit of the Moon about the Earth as the intermediary, neglecting the effect of the Sun altogether. The intermediate orbit adopted by Hill is called the *variational curve* and this must now be defined.

When the solar eccentricity (e') and the solar parallax are neglected, $\Omega = 0$. Also, when the Moon's latitude is neglected, $z = 0$. Equations (4) and (5) then become

$$\left.\begin{aligned} D^2(us) - Du\,.\,Ds + 2\mathrm{m}\,(s\,Du - u\,Ds) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 &= C \\ D\,(u\,Ds - s\,Du - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) &= 0 \end{aligned}\right\} \quad \ldots\ldots(6)$$

which must be equivalent to (1), whence in fact they can be directly deduced. The constant κ (or μ) has been eliminated and the constant C has been introduced. There must be a relation between them which can be found by reference to the original equations of motion. Hill's variational curve is defined as that particular solution of (1) or (6) which represents a periodic orbit. Since the axes of reference rotate at the rate n' the period of this orbit must be $2\pi/(n - n')$ where n is the mean motion of the Moon. From this it follows that the coordinates X, Y of the solution have this period and can be expressed in the form of Fourier series in $(n - n')\,t$, while u, s can be expressed in the form of Laurent series in ζ. The coefficients will be developed in powers of m, and this is an essential advantage of the method, since it is precisely this development which is less easy by the earlier methods. As a particular solution of the equations the symmetrical periodic orbit involves no arbitrary constants beyond those already introduced, namely n, which depends on the actual scale of the lunar orbit, and t_0, which gives an arbitrary epoch corresponding with the fact that (6) do not involve the independent variable explicitly.

The existence of such periodic orbits is assumed. The question has been discussed analytically by Poincaré (*Méthodes Nouvelles*, Tome I), who has proved that they do exist in general. To some extent the assumption will be found practically justified by the results. But there is no doubt on the point. The periodic orbit in the actual circumstances could be found by the method of quadratures.

227. The assumption that the periodic orbit required is symmetrical about both axes at once limits the form of the expansions. For with this limitation X, Y must be of the form

$$X = \sum_0^\infty A_{2i+1} \cos(2i+1)\,\xi, \quad Y = \sum_0^\infty A'_{2i+1} \sin(2i+1)\,\xi, \quad \xi = (n - n')(t - t_0)$$

where $Y = 0$ when $t = t_0$. Hence

$$u = \sum_0^\infty \{\tfrac{1}{2}(A_{2i+1} + A'_{2i+1})\,\zeta^{2i+1} + \tfrac{1}{2}(A_{2i+1} - A'_{2i+1})\,\zeta^{-2i-1}\} = \mathbf{a} \sum_{-\infty}^{\infty} a_{2i}\,\zeta^{2i+1}$$

$$s = \sum_0^\infty \{\tfrac{1}{2}(A_{2i+1} - A'_{2i+1})\,\zeta^{2i+1} + \tfrac{1}{2}(A_{2i+1} + A'_{2i+1})\,\zeta^{-2i-1}\} = \mathbf{a} \sum_{-\infty}^{\infty} a_{-2i-2}\,\zeta^{2i+1}$$

where

$$A_{2i+1} = \mathbf{a}\,(a_{2i} + a_{-2i-2}), \quad A'_{2i+1} = \mathbf{a}\,(a_{2i} - a_{-2i-2}).$$

As it is necessary to multiply such series together and to exhibit the products as double summations, it is convenient to write

$$\left.\begin{aligned} u &= \mathbf{a}\,\Sigma a_{2i}\,\zeta^{2i+1} &&= \mathbf{a}\,\sum_i a_{2j-2i-2}\,\zeta^{2j-2i-1} \\ s &= \mathbf{a}\,\Sigma a_{-2i-2}\,\zeta^{2i+1} &&= \mathbf{a}\,\sum_i a_{-2j+2i}\,\zeta^{2j-2i-1} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$Du = \mathbf{a}\,\Sigma\,(2i+1)\,a_{2i}\,\zeta^{2i+1} = \mathbf{a}\,\sum_i (2j - 2i - 1)\,a_{2j-2i-2}\,\zeta^{2j-2i-1}$$

$$Ds = \mathbf{a}\,\Sigma\,(2i+1)\,a_{-2i-2}\,\zeta^{2i+1} = \mathbf{a}\,\sum_i (2j - 2i - 1)\,a_{-2j+2i}\,\zeta^{2j-2i-1}$$

or similar equivalent forms, so as to retain always a fixed coefficient a_{2i} and a fixed power ζ^{2j} in the typical constituent. The result of substituting the series in (6) is:

$$\begin{aligned} \mathbf{a}^{-2}C = {} & \sum_i\sum_j 4j^2 a_{2i}\,a_{-2j+2i}\,\zeta^{2j} - \sum_i\sum_j (2i+1)(2j-2i-1)\,a_{2i}\,a_{-2j+2i}\,\zeta^{2j} \\ & + 2\mathrm{m}\sum_i\sum_j (4i + 2 - 2j)\,a_{2i}\,a_{-2j+2i}\,\zeta^{2j} \\ & + \tfrac{9}{4}\mathrm{m}^2 \sum_i\sum_j a_{2i}\,(2a_{-2j+2i} + a_{2j-2i-2} + a_{-2j-2i-2})\,\zeta^{2j} \end{aligned}$$

$$\begin{aligned} 0 = {} & \sum_i\sum_j 2j\,(2j - 4i - 2)\,a_{2i}\,a_{-2j+2i}\,\zeta^{2j} - 2\mathrm{m}\sum_i\sum_j 2j a_{2i}\,a_{-2j+2i}\,\zeta^{2j} \\ & + \tfrac{9}{2}\mathrm{m}^2 \sum_i\sum_j a_{2i}\,(a_{2j-2i-2} - a_{-2j-2i-2})\,\zeta^{2j} \end{aligned}$$

where i and j have all positive and negative integral values. The coefficients of every power of ζ must vanish identically, and therefore

$$\mathbf{a}^{-2}C = \sum_i \{(2i+1)^2 + 4\mathrm{m}\,(2i+1) + \tfrac{9}{2}\mathrm{m}^2\}\,a^2_{2i} + \tfrac{9}{2}\mathrm{m}^2 \sum_i a_{2i}\,a_{-2i-2} \quad\ldots(8)$$

when $j = 0$, and

$$\begin{aligned} 0 = {} & \sum_i \{4j^2 + (2i+1)(2i+1-2j) + 4\mathrm{m}\,(2i+1-j) + \tfrac{9}{2}\mathrm{m}^2\}\,a_{2i}\,a_{-2j+2i} \\ & + \tfrac{9}{4}\mathrm{m}^2 \sum_i a_{2i}\,(a_{2j-2i-2} + a_{-2j-2i-2}) \end{aligned}$$

$$0 = -\sum_i 4j\,(2i + 1 - j + \mathrm{m})\,a_{2i}\,a_{-2j+2i} + \tfrac{9}{2}\mathrm{m}^2 \sum_i a_{2i}\,(a_{2j-2i-2} - a_{-2j-2i-2})$$

when j has any other value.

228. Owing to the introduction of **a**, one coefficient a_0 may be made equal to 1, though retained for the sake of symmetry. Then, if m is a small quantity of the first order, a_p is found to be of order $|p|$, being a function of m alone. This fact makes it possible to obtain the coefficients by a process of continued approximation, provided m is sufficiently small. The terms containing $a_0 a_{2j}$, $a_0 a_{-2j}$ in the last equations are obtained when $i = j$ and $i = 0$, and they are respectively

$$\{4j^2 + 2j + 1 + 4\mathrm{m}(j+1) + \tfrac{9}{2}\mathrm{m}^2\} a_0 a_{2j} + \{4j^2 - 2j + 1 - 4\mathrm{m}(j-1) + \tfrac{9}{2}\mathrm{m}^2\} a_0 a_{-2j}$$

and

$$-4j(1 + j + \mathrm{m}) a_0 a_{2j} - 4j(1 - j + \mathrm{m}) a_0 a_{-2j} \quad \ldots\ldots\ldots\ldots\ldots(9)$$

Let the two equations be combined so as to eliminate the second of these terms. The result may be written:

$$\sum_i a_{2i} \{[2j, 2i]\, a_{-2j+2i} + [2j, +]\, a_{2j-2i-2} + [2j, -]\, a_{-2j-2i-2}\} = 0 \quad \ldots(10)$$

where

$$[2j, 2i] = -\frac{i}{j} \cdot \frac{8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2 + 4(i-j)(j - 1 - \mathrm{m})}{8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2}$$

$$[2j, +] = -\frac{3\mathrm{m}^2}{16j^2} \cdot \frac{4j^2 - 8j - 2 - 4\mathrm{m}(j+2) - 9\mathrm{m}^2}{8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2}$$

$$[2j, -] = -\frac{3\mathrm{m}^2}{16j^2} \cdot \frac{20j^2 - 16j + 2 - 4\mathrm{m}(5j - 2) + 9\mathrm{m}^2}{8j^2 - 2 - 4\mathrm{m} + \mathrm{m}^2}$$

the common divisor being chosen so that the coefficient of $a_0 a_{2j}$, $[2j, 2j]$, is -1, while $[2j, 0] = 0$.

If, on the other hand, the term in $a_0 a_{2j}$ be eliminated, the result will be found to be

$$\sum_i a_{2i} \{[-2j, 2i - 2j]\, a_{-2j+2i} + [-2j, +]\, a_{-2j-2i-2} + [-2j, -]\, a_{2j-2i-2}\} = 0$$

which can be deduced from the same series of equations (10) by changing the sign of j and then writing $i - j$ for i in the first term. This single series is therefore sufficient. The last equation can also be written

$$\sum_i \{[-2j, -2i]\, a_{2j-2i} a_{-2i} + [-2j, -]\, a_{2j-2i-2} a_{2i} + [-2j, +]\, a_{-2j-2i-2} a_{2i}\} = 0$$

and hence the rule for connecting the pair of equations corresponding to $\pm j$: in terms multiplied by $[2j, 2i]$ change the signs of j and i throughout (both in coefficients and in suffixes); in the other terms write $[-2j, -]$ for $[2j, +]$ and $[-2j, +]$ for $[2j, -]$, the suffixes being unchanged.

229. Since the coefficients $[2j, \pm]$ are of the second order in m, the orders of the three terms are respectively

$$2|i| + 2|i - j|, \quad 2|i| + 2|i + 1 - j| + 2, \quad 2|i| + 2|i + 1 + j| + 2$$

which are at least

$$2|j|, \quad 2|j - 1| + 2, \quad 2|j + 1| + 2.$$

Let the equations be written down so as to include all quantities of the sixth order (neglecting m^8). This requires $j = \pm 1, \pm 2, \pm 3$. The orders of the terms with the only possible values of i are:

$j = 1$, $i = 2\,(6, 10, 14), 1\,(2, 6, 10), 0\,(2, 2, 6), -1\,(6, 6, 6), -2\,(10, 10, 6)$

$j = 2$, $i = 2\,(4, 8, 16), 1\,(4, 4, 12), 0\,(4, 4, 8)$

$j = 3$, $i = 3\,(6, 10, 22), 2\,(6, 6, 18), 1\,(6, 6, 14), 0\,(6, 6, 10)$.

Hence the required equations are:

$$a_0 a_2 = [2, 4]\,a_2 a_4 + [2, -2]\,a_{-2}a_{-4} + [2, +]\,(2a_2 a_{-2} + a_0^2) + [2, -]\,(2a_0 a_{-4} + a^2_{-2})$$

$$a_0 a_{-2} = [-2, -4]\,a_{-2}a_{-4} + [-2, 2]\,a_2 a_4 + [-2, -]\,(2a_2\,a_{-2} + a_0^2) + [-2, +]\,(2a_0 a_{-4} + a^2_{-2})$$

$$a_0 a_4 = [4, 2]\,a_2 a_{-2} + [4, +]\,2a_0 a_2$$

$$a_0 a_{-4} = [-4, -2]\,a_2 a_{-2} + [-4, -]\,2a_0 a_2$$

$$a_0 a_6 = [6, 4]\,a_{-2}a_4 + [6, 2]\,a_2 a_{-4} + [6, +]\,(2a_0 a_4 + a_2^2)$$

$$a_0 a_{-6} = [-6, -4]\,a_2 a_{-4} + [-6, -2]\,a_{-2}a_4 + [-6, -]\,(2a_0 a_4 + a_2^2).$$

Thus, since $a_0 = 1$, if m^6 be neglected,

$$a_2 = [2, +], \quad a_{-2} = [-2, -]$$

and then, neglecting m^8,

$$a_4 = [4, 2]\,[2, +]\,[-2, -] + 2\,[4, +]\,[2, +]$$
$$a_{-4} = [-4, -2]\,[2, +]\,[-2, -] + 2\,[-4, -]\,[2, +].$$

These values will give a_6, a_{-6} as far as m^9, and inserted on the right-hand side of the first pair of equations they give second approximations to a_2, a_{-2} of the same order. It is to be noticed that each stage of further development carries an equation four orders higher.

The ratio of the mean motions of the Sun and Moon, and therefore the numerical value of m, is known with great accuracy from observation. Hill adopted the value

$$\mathrm{m} = n'/(n - n') = 0{\cdot}08084\;89338\;08312.$$

Hence it is practicable to introduce the numerical value of m from the beginning, and the approximation to great accuracy in the calculation of $a_{\pm 2}, \ldots$ is then extremely rapid by the above method. This is the process which has been adopted in the latest form of lunar theory. It is also possible by giving m other values to trace the development of the whole family of periodic orbits of lunar type. These orbits are of great theoretical interest, especially for larger values of m. But it is evident that the effect of the neglected parallactic terms will become more considerable, and such results may differ sensibly from true solutions of the restricted problem of three bodies. Also when m exceeds $\frac{1}{3}$ the question of convergence begins to introduce practical difficulties and the method of quadratures, followed by Sir G. H. Darwin and others, becomes necessary.

230. To find the value of $\mathbf{a}$ recourse must be had to an equation of motion which has not been reduced to a homogeneous form in u, s. Since $\Omega = z = 0$ and $r^2 = us$, the first of (2) becomes in the present case

$$(D^2 + 2\mathrm{m}D + \tfrac{9}{2}\mathrm{m}^2)\, u + \tfrac{9}{2}\mathrm{m}^2 s = \kappa u\, (us)^{-\frac{3}{2}}$$

or

$$\mathbf{a} \sum_i \{(2i+1)^2 + 2\mathrm{m}\,(2i+1) + \tfrac{3}{2}\mathrm{m}^2\}\, a_{2i}\zeta^{2i+1} + \tfrac{3}{2}\mathrm{m}^2\mathbf{a} \sum_i a_{2i}\zeta^{-2i-1} = \kappa u\, (us)^{-\frac{3}{2}}.$$

This equation must hold for all values of ζ, including $\zeta = 1$. Then $u = s = \mathbf{a} \sum a_{2i}$, and therefore

$$\mathbf{a} \sum \{(2i + 1 + \mathrm{m})^2 + 2\mathrm{m}^2\}\, a_{2i} = \kappa \mathbf{a}^{-2} (\sum a_{2i})^{-2}.$$

But (§ 224) $\kappa = \mu\,(n - n')^{-2} = \mu\,(1 + \mathrm{m})^2 n^{-2}$, so that

$$n^2\mathbf{a}^3 = \mu\,(1 + \mathrm{m})^2 (\sum a_{2i})^{-2} [\sum \{(2i + 1 + \mathrm{m})^2 + 2\mathrm{m}^2\}\, a_{2i}]^{-1} \quad \text{......(11)}$$

It has been usual to write $n^2a^3 = \mu$, a being the mean distance which would correspond to the mean motion n in the absence of solar or other perturbations. Thus $\mathbf{a} = a\,(1 + \text{powers of m})$ when the values of a_{2i} are inserted. The precise form of this relation is required only when it is desired to compare two theories expressed in terms of $\mathbf{a}$ and a respectively. The constant $\mathbf{a}$ fixes the scale of the orbit and therefore depends on the parallax, which is observed directly.

When the coefficients a_{2i} and $\mathbf{a}$ have been determined, (8) gives the value of C, if it be required.

For the transformation to polar coordinates,

$$r\cos(v - nt - \epsilon) = r\cos(v - n't - \epsilon' - \xi) = X\cos\xi + Y\sin\xi = \tfrac{1}{2}(u\zeta^{-1} + s\zeta)$$

$$r\sin(v - nt - \epsilon) = r\sin(v - n't - \epsilon' - \xi) = Y\cos\xi - X\sin\xi = \tfrac{1}{2}(s\zeta - u\zeta^{-1})\,\iota$$

where $\epsilon = \epsilon' - (n - n')\, t_0$, since $\xi = (n - n')(t - t_0)$ and $\iota\xi = \log\zeta$. Hence

$$\left.\begin{aligned} r\cos(v - nt - \epsilon) &= \mathbf{a}\,\{1 + (a_2 + a_{-2})\cos 2\xi + (a_4 + a_{-4})\cos 4\xi + \dots\} \\ r\sin(v - nt - \epsilon) &= \mathbf{a}\,\{\quad (a_2 - a_{-2})\sin 2\xi + (a_4 - a_{-4})\sin 4\xi + \dots\} \end{aligned}\right\} \text{...(12)}$$

which lead to the determination of r and v, the more simply because $v - nt - \epsilon$ is evidently of the second order in m.

231. The use of rectangular coordinates is a distinctive feature of Hill's method. But for some purposes polar coordinates present advantages. By a simple change of units and notation (1) become

$$\frac{d^2p}{dt^2} - 2\frac{dq}{dt} = 3p - \frac{p}{r^3}$$

$$\frac{d^2q}{dt^2} + 2\frac{dp}{dt} = -\frac{q}{r^3}$$

which can be reduced to canonical form by putting (cf. § 216)

$$p' = \dot{p} - q, \quad q' = \dot{q} + p$$

$$H = \tfrac{1}{2}(p' + q)^2 + \tfrac{1}{2}(q' - p)^2 - \tfrac{3}{2}p^2 - r^{-1}.$$

The transformation to new variables, r, l; r', l', defined by

$$p = r\cos l, \quad p' = r'\cos l - r^{-1}l'\sin l$$
$$q = r\sin l, \quad q' = r'\sin l + r^{-1}l'\cos l$$

will leave the canonical form unchanged, since

$$p'dp + q'dq - (r'dr + l'dl) \equiv 0$$

and therefore it is an extended point transformation (§ 125). Let t be eliminated from the equations by taking l as the independent variable. After writing out the equations in explicit form make the transformation

$$r = 1/\sigma, \quad r' = \rho/\sigma, \quad l' = \omega/\sigma^2$$

and finally put $\epsilon = \sigma^3$. The result is to give the equations

$$(\omega - 1)\frac{d\epsilon}{dl} = -3\rho\epsilon$$

$$(\omega - 1)\frac{d\rho}{dl} = \omega^2 - \rho^2 + \tfrac{3}{2}\cos 2l + \tfrac{1}{2} - \epsilon$$

$$(\omega - 1)\frac{d\omega}{dl} = -2\rho\omega - \tfrac{3}{2}\sin 2l$$

and the integral $H = h$ becomes

$$\tfrac{1}{2}\rho^2 + \tfrac{1}{2}(\omega - 1)^2 - \tfrac{3}{2}\cos^2 l - (h\epsilon^{\frac{2}{3}} + \epsilon) = 0.$$

Assume a solution in the form

$$\rho = \iota\sum_{-\infty}^{\infty} a_{2n}e^{2\iota nl/k}, \quad \omega = \sum_{-\infty}^{\infty} b_{2n}e^{2\iota nl/k}, \quad \epsilon = \sum_{-\infty}^{\infty} c_{2n}e^{2\iota nl/k}.$$

For a periodic orbit described always in one direction as regards l these series are convergent, and if the coefficients are real, $a_{2n} = -a_{-2n}$, $b_{2n} = b_{-2n}$, $c_{2n} = c_{-2n}$, and therefore

$$\rho = \frac{1}{r}\frac{dr}{dt} = -2\sum_{1}^{\infty} a_{2n}\sin\frac{2nl}{k}$$

$$\omega = 1 + \frac{dl}{dt} = b_0 + 2\sum_{1}^{\infty} b_{2n}\cos\frac{2nl}{k}$$

$$\epsilon = \frac{1}{r^3} = c_0 + 2\sum_{1}^{\infty} c_{2n}\cos\frac{2nl}{k}.$$

The index k is arbitrary. It may be proved that if k is an odd integer the orbit is completed in k circuits and is symmetrical about both axes, and if k is an even integer the orbit is completed in $\tfrac{1}{2}k$ circuits and is symmetrical about the axis of p only. For Hill's variational curve $k = 1$.

The substitution of the assumed series in the equations leads to three series of equations which must be solved by continued approximation as in

Hill's method. A most interesting result is that the series for ϵ converges with exceptional rapidity, so that the equation

$$r^{-3} = c_0 + 2c_2 \cos 2l$$

where $c_0 = 93c_2$ nearly, represents the variational curve with an error which on the scale of the lunar orbit is no more than half a mile. No simpler idea of the nature of this curve could possibly be given.

It may be left as an exercise to the student to fill in the details of the outline conveyed in this section*.

232. The method by which the variational curve can be determined with any required degree of accuracy has been fully explained. But it must not be supposed that this curve represents the lunar orbit in any true sense. It is merely a particular solution of equations which are themselves only a degenerate form of those which characterize the Moon's motion, and the only significant parameter involved is the mean motion of the Moon. The next step is to seek the form of the general solution of the same equations. With this object it is necessary to study the variation of the particular solution and to determine a fundamental quantity c.

With some change of notation (3) and (4) of § 214 give

$$\frac{d^2}{dt^2}\delta N + \Theta\,\delta N = 0 \quad\text{..........................(13)}$$

where, in the application to (1),

$$\Theta = 2n'^2 + 2(\dot\psi + n')^2 - \nabla^2 F + \frac{1}{V}\frac{d^2V}{dt^2}, \quad F = \mu r^{-1} + \tfrac{3}{2}n'^2 X^2$$

δN being the normal displacement to the variational curve, ψ the inclination of the tangent to the axis of X, and V the relative velocity. In terms of u, s,

$$V^2 = \dot X^2 + \dot Y^2 = \dot u \dot s = -\nu^2 Du\, Ds$$

since $d/dt = \iota\nu D$. Hence, R being the radius of curvature,

$$\dot\psi = V/R = (\ddot Y \dot X - \ddot X \dot Y)/V^2 = \tfrac{1}{2}\iota(\ddot s \dot u - \ddot u \dot s)/V^2 = \tfrac{1}{2}\nu\left(\frac{D^2u}{Du} - \frac{D^2s}{Ds}\right).$$

Also

$$\frac{1}{V}\frac{d^2V}{dt^2} = \frac{1}{V}\frac{d}{dt}\left(\frac{1}{2V}\frac{dV^2}{dt}\right) = \frac{d}{dt}\left(\frac{1}{2V^2}\frac{dV^2}{dt}\right) + \frac{1}{4V^4}\left(\frac{dV^2}{dt}\right)^2$$

$$= -\nu^2 D\left(\frac{DV^2}{2V^2}\right) - \nu^2\left(\frac{DV^{2\,2}}{2V^2}\right)$$

$$= -\tfrac{1}{2}\nu^2 D\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right) - \tfrac{1}{4}\nu^2\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right)^2.$$

* Cf. J. F. Steffensen, Royal Danish Academy, *Forhandlinger* (1909).

Finally

$$\nabla^2 F = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) F = \frac{\partial^2}{\partial X^2}(\tfrac{3}{2}n'^2 X^2) - \mu\left(\frac{\partial^2 r^{-1}}{\partial z^2}\right)_{z=0} = \mu/r^3 + 3n'^2.$$

Therefore, since $\nu = n' - n$, $n' = \mathrm{m}\nu$ and $\mu = \kappa\nu^2$,

$$\nu^{-2}\Theta = -\kappa/r^3 - \mathrm{m}^2 + 2\left\{\tfrac{1}{2}\left(\frac{D^2u}{Du} - \frac{D^2s}{Ds}\right) + \mathrm{m}\right\}^2$$

$$- \tfrac{1}{2} D\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right) - \tfrac{1}{4}\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right)^2 \quad\ldots\ldots\ldots\ldots(14)$$

Now since $u = \zeta\Sigma a_{2i}\zeta^{2i}$, $s = \zeta^{-1}\Sigma a_{2i}\zeta^{-2i}$ and $D = \zeta d/d\zeta = -\zeta^{-1} d/d\zeta^{-1}$,

$$D^2u/Du = \sum_i U_i\zeta^{2i}, \quad D^2s/Ds = -\sum_i U_i\zeta^{-2i}$$

and U_i can be calculated by equating coefficients in

$$\sum_i (2i+1)^2 a_{2i}\zeta^{2i+1} = \sum_i (2i+1) a_{2i}\zeta^{2i+1} \cdot \sum_i U_i\zeta^{2i}.$$

Similarly, by the first of (2) when $\Omega = 0$,

$$u(\kappa r^{-3} + \mathrm{m}^2) = 2u\sum_i M_i\zeta^{2i} = D^2u + 2\mathrm{m}\, Du + \tfrac{1}{2}\mathrm{m}^2(5u + 3s)$$

so that

$$2\sum_i a_{2i}\zeta^{2i+1} \cdot \sum_i M_i\zeta^{2i} = \sum_i \{(2i+1)^2 + 2\mathrm{m}(2i+1) + \tfrac{5}{2}\mathrm{m}^2\} a_{2i}\zeta^{2i+1} + \tfrac{3}{2}\mathrm{m}^2\Sigma a_{-2i-2}\zeta^{2i+1}$$

whence M_i can be calculated in the same way. When U_i, M_i have been found it remains to substitute the series in (14), a process which involves squaring two series, and the result may be written in the form

$$\nu^{-2}\Theta = \sum_i \Theta_i\zeta^{2i}.$$

Thus (13) becomes

$$D^2\delta N = (\sum_i \Theta_i\zeta^{2i})\,\delta N \quad\ldots\ldots\ldots\ldots(15)$$

and the derivation of Θ_i has been fully explained. It is easily seen that $\Theta_{-i} = \Theta_i$ and that M_i, U_i and Θ_i are of the order $|2i|$ in m.

233. Owing to the symmetry of the variational curve Θ is a periodic function with the half period of the curve, $\pi/(n - n')$. Hence by § 215 one solution of (15) has the form

$$\delta N = \zeta^{\mathrm{c}}\Sigma b_i\zeta^{2i}$$

and c is the quantity which is now required. The result of substituting this series is

$$\sum_j b_j(\mathrm{c} + 2j)^2\zeta^{\mathrm{c}+2j} = \sum_i\sum_j \Theta_i b_{j-i}\zeta^{\mathrm{c}+2j}$$

which must be an identity, and therefore for every value of j

$$b_j(\mathrm{c} + 2j)^2 = \sum_i \Theta_i b_{j-i}$$

or more fully, since $\Theta_i = \Theta_{-i}$,

$$\ldots - \Theta_2 b_{j-2} - \Theta_1 b_{j-1} + \{(\mathrm{c} + 2j)^2 - \Theta_0\} b_j - \Theta_1 b_{j+1} - \Theta_2 b_{j+2} - \ldots = 0.$$

These equations are of infinite order. Nevertheless, let the coefficients b_i be eliminated in the same way as though their number were finite. Then $\Delta(c) = 0$ where $\Delta(c)$ represents the determinant of infinite order

$$\left|\begin{array}{ccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots, & \dfrac{(c-4)^2-\Theta_0}{4^2-\Theta_0}, & \dfrac{-\Theta_1}{4^2-\Theta_0}, & \dfrac{-\Theta_2}{4^2-\Theta_0}, & \dfrac{-\Theta_3}{4^2-\Theta_0}, & \dfrac{-\Theta_4}{4^2-\Theta_0}, & \ldots \\ \ldots, & \dfrac{-\Theta_1}{2^2-\Theta_0}, & \dfrac{(c-2)^2-\Theta_0}{2^2-\Theta_0}, & \dfrac{-\Theta_1}{2^2-\Theta_0}, & \dfrac{-\Theta_2}{2^2-\Theta_0}, & \dfrac{-\Theta_3}{2^2-\Theta_0}, & \ldots \\ \ldots, & \dfrac{-\Theta_2}{0^2-\Theta_0}, & \dfrac{-\Theta_1}{0^2-\Theta_0}, & \dfrac{c^2-\Theta_0}{0^2-\Theta_0}, & \dfrac{-\Theta_1}{0^2-\Theta_0}, & \dfrac{-\Theta_2}{0^2-\Theta_0}, & \ldots \\ \ldots, & \dfrac{-\Theta_3}{2^2-\Theta_0}, & \dfrac{-\Theta_2}{2^2-\Theta_0}, & \dfrac{-\Theta_1}{2^2-\Theta_0}, & \dfrac{(c+2)^2-\Theta_0}{2^2-\Theta_0}, & \dfrac{-\Theta_1}{2^2-\Theta_0}, & \ldots \\ \ldots, & \dfrac{-\Theta_4}{4^2-\Theta_0}, & \dfrac{-\Theta_3}{4^2-\Theta_0}, & \dfrac{-\Theta_2}{4^2-\Theta_0}, & \dfrac{-\Theta_1}{4^2-\Theta_0}, & \dfrac{(c+4)^2-\Theta_0}{4^2-\Theta_0}, & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|$$

each row being divided by such a factor that the constituent in the leading diagonal becomes 1 when $c = 0$. This is Hill's celebrated determinant, which introduced the consideration of the meaning and convergence* of determinants of infinite order into mathematical analysis.

234. The determinant $\Delta(-c) = \Delta(c)$, for the change only reverses the order of the constituents in the leading diagonal. Also $\Delta(c + 2j) = \Delta(c)$, for the displacement of the leading diagonal along itself may be compensated by moving the divisors of the rows. Hence if c_0 is a root of $\Delta(c)$, $\pm c_0 + 2j$ are also roots. The highest power of c in the development is given by the product of terms in the leading diagonal, and this product is

$$\Delta_0(c) = \prod_{-\infty}^{\infty} \frac{(c+2j)^2 - \Theta_0}{4j^2 - \Theta_0} = \prod_{-\infty}^{\infty} \frac{c^2 - (2j + \sqrt{\Theta_0})^2}{(2j + \sqrt{\Theta_0})^2}$$

$$= (\cos \pi c - \cos \pi \sqrt{\Theta_0}) / (1 - \cos \pi \sqrt{\Theta_0}).$$

It follows that

$$\Delta(c) = (\cos \pi c - \cos \pi c_0) / (1 - \cos \pi \sqrt{\Theta_0})$$

for this contains the right number of roots, the same as $\Delta_0(c)$, and the same coefficient of the highest power of c. The roots are those already found, and there are no others. But this equation shows that

$$\Delta(0) = (1 - \cos \pi c_0) / (1 - \cos \pi \sqrt{\Theta_0})$$

and therefore c_0 is a root of

$$\sin^2 \tfrac{1}{2}\pi c_0 = \Delta(0) \sin^2 \tfrac{1}{2}\pi \sqrt{\Theta_0} \quad \text{.......................(16)}$$

* Cf. Whittaker's *Modern Analysis*, p. 35; Whittaker and Watson, p. 36.

The solution of $\Delta(c)=0$ is thus reduced to the calculation of $\Delta(0)$. The latter determinant is convergent if $\Sigma_i \Theta_i$ is convergent, and this may be assumed for sufficiently small values of m.

As a matter of fact in the present case $\Delta(0)$ is not only convergent but very rapidly convergent. It may be written in the form

$$\Delta(0)=\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots, & 1, & -\beta_j\Theta_1, & -\beta_j\Theta_2, & \cdots, & \cdots \\ \cdots, & -\beta_{j-1}\Theta_1, & 1, & -\beta_{j-1}\Theta_1, & -\beta_{j-1}\Theta_2, & \cdots \\ \cdots, & -\beta_{j-2}\Theta_2, & -\beta_{j-2}\Theta_1, & 1, & -\beta_{j-2}\Theta_1, & \cdots \\ \cdots, & \cdots, & -\beta_{j-3}\Theta_2, & -\beta_{j-3}\Theta_1, & 1, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

where

$$\beta_j = 1/(4j^2-\Theta_0).$$

Suppose every Θ_j to be multiplied by θ^j. If then the sign of θ be changed the sign of every alternate constituent in every row and every column is changed. Multiply every alternate row and every alternate column by -1 and the original determinant is restored. This involves multiplication of $\Delta(0,-\theta)$ by an even power of -1, since the number of rows and columns is equal. Hence $\Delta(0,-\theta)=\Delta(0,\theta)$, and $\Delta(0,\theta)$ is an even function of θ. But the power of θ in any term of the development of $\Delta(0,\theta)$ is the sum of the suffixes of the Θ_j associated with it. Therefore the sum of the suffixes in any term of the development of $\Delta(0)$ is even. Since Θ_j is of the order $|2j|$ in m, this means that the order of every term is a multiple of 4.

It is evident that the determinant $\Delta(0)$ must be developed axially, the term of zero order, 1, coming from the leading diagonal alone. There can be no term in Θ_j alone, for Θ_j incapacitates by its row and column two units from the leading diagonal as cofactors. Similarly a product $\Theta_i\Theta_j$ incapacitates more than two such units unless their rows and columns intersect on the leading diagonal. Thus $i=j$ and the only terms of binary type involve squares.

235. The mode of developing $\Delta(0)$ will be sufficiently understood if m^{12} be neglected. The sum of the suffixes can only be 0, 2 or 4. Hence the only possible terms are of the type

$$\Delta(0)=1+A\Theta_1^2+B\Theta_2^2+C\Theta_1^2\Theta_2+D\Theta_1^4.$$

It is also easy to see how each of these terms arises. Thus

$$A\Theta_1^2=\sum_j\begin{vmatrix} 0, & -\beta_j\Theta_1 \\ -\beta_{j-1}\Theta_1, & 0\end{vmatrix}, \quad B\Theta_2^2=\sum_j\begin{vmatrix} 0, & -\beta_j\Theta_2 \\ -\beta_{j-2}\Theta_2, & 0\end{vmatrix}$$

$$A=-\sum_j\beta_j\beta_{j-1}, \quad B=-\sum_j\beta_j\beta_{j-2}.$$

The next term corresponds to three consecutive diagonal constituents, and

$$C\Theta_1^2\Theta_2 = \sum_j \begin{vmatrix} 0 & , -\beta_j\Theta_1 & , -\beta_j\Theta_2 \\ -\beta_{j-1}\Theta_1, & 0 & , -\beta_{j-1}\Theta_1 \\ -\beta_{j-2}\Theta_2, & -\beta_{j-2}\Theta_1, & 0 \end{vmatrix} = -2\sum_j \beta_j\beta_{j-1}\beta_{j-2}\,\Theta_1^2\Theta_2.$$

Finally, the term in Θ_1^4 must correspond to four diagonal constituents only and it is therefore

$$D\Theta_1^4 = \sum_i\sum_j \begin{vmatrix} 0 & , -\beta_i\Theta_1 \\ -\beta_{i-1}\Theta_1, & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & , -\beta_j\Theta_1 \\ -\beta_{j-1}\Theta_1, & 0 \end{vmatrix}$$

$$D = \sum_i\sum_j \beta_i\beta_{i-1}\beta_j\beta_{j-1} = A^2 - \sum_j \beta_j^2\beta^2_{j-1} - 2\sum_j \beta_{j+1}\beta_j^2\beta_{j-1}$$

for, as the two minors must not overlap, i cannot have the values j or $j \pm 1$.

It remains to calculate the values of these coefficients. Let $\Theta_0 = 4\alpha^2$. Then

$$\sum_j \beta_j\beta_{j-1} = \sum_j \frac{1}{16(\alpha^2 - j^2)\{\alpha^2 - (j-1)^2\}}$$

$$= \sum_j \frac{1}{32\alpha(2\alpha-1)}\left(\frac{1}{\alpha-j} + \frac{1}{\alpha+j-1}\right) - \sum_j \frac{1}{32\alpha(2\alpha+1)}\left(\frac{1}{\alpha+j} + \frac{1}{\alpha-j+1}\right)$$

$$= \sum_{-\infty}^{\infty} \frac{1}{8\alpha(4\alpha^2-1)} \cdot \frac{1}{\alpha+j} = \frac{1}{8\alpha(4\alpha^2-1)}\left\{\frac{1}{\alpha} + 2\alpha\sum_1^{\infty}\frac{1}{\alpha^2-j^2}\right\}$$

$$= \frac{\pi\cot\pi\alpha}{8\alpha(4\alpha^2-1)} = \frac{\pi\cot\frac{1}{2}\pi\sqrt{\Theta_0}}{4\sqrt{\Theta_0}(\Theta_0-1)}.$$

The other coefficients can be calculated similarly by first reducing to the form of partial fractions. Hill's results include all terms of order less than 16, and with the value of m already given (§ 229) he obtained the value

$$c_0 = 1{\cdot}07158\ 32774\ 16012.$$

Without going further than the term of which the form has actually been found here,

$$\Delta(0) = 1 + \tfrac{1}{4}\pi\Theta_1^2\cot\tfrac{1}{2}\pi\sqrt{\Theta_0}/(1-\Theta_0)\sqrt{\Theta_0} \quad \text{............(17)}$$

The argument given above as to the order of the terms refers to Θ_1, Θ_2, ... and not to effects arising from Θ_0. But $1 - \Theta_0$ is itself of the first order, and therefore this expression neglects m^7 instead of m^8. Since $m = 0{\cdot}08$ the error in c_0 might be expected to occur at about the seventh decimal place, and in fact it is about 5 units in this place. This simple expression, involving only Θ_0 and Θ_1, is therefore very approximate.

It may be noticed that $\pm\iota c(n - n')$ are the characteristic exponents of the variational curve. Since c is real this curve represents a stable orbit for small variations.

236. The introduction of the eliminant of infinite order was a bold and original expedient on the part of Hill, though justified later by analysis. But an analogous method had been used earlier by Adams, whose results were published after the appearance of Hill's. They refer to the integration of the third equation of (2) when $\Omega = 0$, or

$$D^2 z - z(\kappa r^{-3} + \mathrm{m}^2) = 0.$$

If z be neglected in the coefficient of z, that is in r^{-3}, the series already used in § 232 may be inserted, and the equation becomes

$$D^2 z = (2\sum_i M_i \zeta^{2i})\, z$$

which, since $M_i = M_{-i}$ is of the order $|\,2i\,|$ in m, is of exactly the same form as (15). A solution is known to be of the type

$$z = \zeta^{\mathrm{g}} \sum_i \beta_i \zeta^{2i}$$

and g must be determined from the infinite set

$$\beta_j (\mathrm{g} + 2j)^2 = \sum_i 2M_i \beta_{j-i}.$$

Hence the eliminant is $\Delta'(\mathrm{g}) = 0$, and the solution is given by

$$\sin^2 \tfrac{1}{2}\pi \mathrm{g}_0 = \Delta'(0) \sin^2 \tfrac{1}{2}\pi \sqrt{(2M_0)}$$

where $\Delta'(0)$ is the result of replacing Θ_i by $2M_i$ in $\Delta(0)$.

Adams used the value $m = n'/n = 0{\cdot}0748013$ exactly, which is not quite the same as Hill's value. He thus obtained the corresponding numbers

$$\mathrm{m} = 0{\cdot}08084\ 89030\ 51852, \quad \mathrm{g}_0 = 1{\cdot}08517\ 13927\ 46869.$$

CHAPTER XXI

LUNAR THEORY II

237. It is now necessary to consider the form of the general solution of the equations (6); in the present chapter equations will receive reference numbers in continuation of those assigned in the previous chapter, so that the latter will suffice without referring specifically to the chapter or section in which they occur. The solution of (15) may now be written

$$\delta N = \zeta_1^{\pm c} \Sigma b_i \zeta^{2i}, \quad \log \zeta_1 = \iota (n - n')(t - t_1).$$

The arbitrary constant t_1 makes it possible to assign any required phase to the variation in relation to the periodic solution and as δN is supposed small (so that δN^2 has been neglected) the coefficients b_i may be considered to have a small arbitrary factor. These two arbitraries make the small variation otherwise general. Since c has been determined it would clearly be possible to determine real values of the coefficients (except for the arbitrary factor) by substituting the series in (15), equating coefficients, and proceeding by continued approximation.

Again, if $\delta\sigma$ be the displacement in arc corresponding to δN, by (2) of § 214 adapted to the present notation,

$$2(\dot{\psi} + n')\,\delta N = -V\frac{d}{dt}\left(\frac{\delta\sigma}{V}\right)$$

or (§ 232)

$$\left(\frac{D^2u}{Du} - \frac{D^2s}{Ds} + 2\mathrm{m}\right)\delta N = -\iota V D\left(\frac{\delta\sigma}{V}\right).$$

Hence, V being an even function of ζ, $\iota\delta\sigma$ has the same form as δN. But since

$$V\cos\psi = \dot{X}, \quad V\sin\psi = \dot{Y}$$

$$Ve^{\iota\psi} = \iota\nu Du, \quad Ve^{-\iota\psi} = \iota\nu Ds$$

and

$$\delta N = \delta X \sin\psi - \delta Y\cos\psi = \tfrac{1}{2}\iota\,(\delta u\,.\,e^{-\iota\psi} - \delta s\,.\,e^{\iota\psi})$$

$$\delta\sigma = \delta X\cos\psi + \delta Y\sin\psi = \tfrac{1}{2}\,(\delta u\,.\,e^{-\iota\psi} + \delta s\,.\,e^{\iota\psi})$$

it follows that

$$\delta u = \frac{\nu Du}{V}(\delta N + \iota\delta\sigma), \quad \delta s = \frac{\nu Ds}{V}(\iota\delta\sigma - \delta N).$$

Hence δu, δs, like Du, Ds, are odd functions in ζ with real coefficients, and it is possible to write

$$\delta u = \zeta_1^{\pm c}\zeta\sum_i b_{2i}\zeta^{2i}, \quad \delta s = \zeta_1^{\pm c}\zeta^{-1}\sum_i b_{2i}\zeta^{-2i}$$

the coefficients as expressed being the same in the two series since $\delta u + \delta s = 2\delta X$ is real. For the purpose of this argument it is necessary to associate the $+$ c solution for δu with the $-$ c solution for δs, and to notice that $(\zeta_1/\zeta)^{\pm c}$ are constant conjugate imaginaries with absolute value 1 which have been regarded as external factors of the series with real coefficients for δN, $\iota\delta\sigma$, δu and δs. At the same time $\delta u - \delta s$ is a pure imaginary.

Hence the general solution of (6), differing but little from the variational curve, may be written

$$u = \mathbf{a}\zeta\sum_i\sum_p A_{2i+pc}\zeta^{2i}\zeta_1^{pc}, \quad s = \mathbf{a}\zeta^{-1}\sum_i\sum_p A_{-2i-pc}\zeta^{2i}\zeta_1^{pc}$$

where i has all integral values between $\pm\infty$ and p has the values 0 and ± 1. Also $A_{2i} = a_{2i}$ as in the variational curve and c is a determined function of m which has been denoted by c_0.

238. But the solution which is now sought differs by a finite amount from the variational curve. The above form must therefore be regarded merely as the beginning of the full development. Hence the restriction on p will now be withdrawn and its values will be allowed to range between $\pm\infty$. The coefficients of the first order $A_{2i\pm c}$ contain a small arbitrary parameter e and the higher coefficients $A_{2i\pm pc}$ will be obtained by successive approximation in the ordinary way, so that $A_{2i\pm pc}$ will be of the order $|p|$ in e. The introduction of e into the solution will affect both A_{2i} and c, and a_{2i} and c_0 represent those parts only which are functions of m alone and not of e.

It is assumed that this process will produce convergent series. If they converge they are true solutions of the differential equations, and not otherwise. This recurrent question in dynamical astronomy cannot be dealt with here. But the reader must realize its fundamental importance, and he will understand why so much attention has been given, by Poincaré especially, to discussions of this kind, although they may seem unproductive of new and striking results.

It is now to be noticed that

$$D(\zeta^{2i+1}\zeta_1^{pc}) = (2i+1+pc)\,\zeta^{2i+1}\zeta_1^{pc}, \quad D\zeta^{2i+1+pc} = (2i+1+pc)\,\zeta^{2i+1+pc}$$

and therefore that the result of putting $\zeta_1 = \zeta$ will affect in no way the process of calculating the coefficients. If this substitution is made it is only necessary to retain c explicitly in the index of ζ and to remember that the argument of the trigonometrical term corresponding to $\zeta^{2i+1+pc}$ is

$$(2i+1)(n-n')(t-t_0) + pc\,(n-n')(t-t_1).$$

With this understanding the form of solution becomes

$$u = \mathbf{a}\zeta \sum_i \sum_p A_{2i+pc}\zeta^{2i+pc}, \quad s = \mathbf{a}\zeta^{-1}\sum_i \sum_p A_{-2i-pc}\zeta^{2i+pc} \dots\dots\dots (18)$$

Comparison of these series with (7) shows immediately that the effect of substituting in the differential equations and equating coefficients of ζ^{2j+qc} will follow as before if

$$A, \quad \sum_i \sum_p, \quad 2i+pc, \quad 2j+qc$$

be substituted respectively for

$$a, \quad \sum_i, \quad 2i, \quad 2j.$$

Thus to (10) corresponds the equation

$$\sum_i \sum_p A_{2i+pc}\{[2j+qc,\ 2i+pc]\,A_{-2j+2i-qc+pc}$$
$$+[2j+qc, +]\,A_{2j-2i-2+qc-pc} + [2j+qc, -]\,A_{-2j-2i-2-qc-pc}\} = 0 \quad \dots (19)$$

which holds unless $j = q = 0$. The form of the symbolical coefficients has been given with (10), $[2j+qc,\ 2j+qc] = -1$ is the coefficient of $A_0 A_{2j+qc}$, and $[2j+qc,\ 0] = 0$ is the coefficient of $A_0 A_{-2j-qc}$. The counterpart of (8) is

$$\mathbf{a}^{-2}C = \sum_i \sum_p \{(2i+1+pc)^2 + 4\mathrm{m}(2i+1+pc) + \tfrac{9}{2}\mathrm{m}^2\}\,A^2_{2i+pc}$$
$$+ \tfrac{9}{2}\mathrm{m}^2 \sum_i \sum_p A_{2i+pc}\,A_{-2i-2-pc}.$$

239. Of the first importance are the terms which depend on the first power of the parameter e. When δN^2 was neglected A_{2i} was identical with a_{2i}, and therefore $A_{2i} = a_{2i}$ when e^2 is neglected. Let

$$A_{2i+c} = \mathrm{e}\,\epsilon_i, \quad A_{2i-c} = \mathrm{e}\,\epsilon_i'.$$

The limitation to the first order in e means a return to the equations at the end of § 237 and the only admissible values of q are ± 1. With either value p must be chosen so that c occurs only once in the suffixes of any term, or terms involving e^2 will be introduced. Hence (19) gives

$$\sum_i \{[2j+c,\ 2i+c]\,a_{-2j+2i}\epsilon_i + [2j+c,\ 2i]\,a_{2i}\epsilon'_{-j+i}$$
$$+[2j+c, +]\,(a_{2j-2i-2}\epsilon_i + a_{2i}\epsilon_{j-i-1}) + [2j+c, -]\,(a_{-2j-2i-2}\epsilon_i' + a_{2i}\epsilon'_{-j-i-1})\} = 0$$

$$\sum_i \{[2j-c,\ 2i-c]\,a_{-2j+2i}\epsilon_i' + [2j-c,\ 2i]\,a_{2i}\epsilon_{-j+i}$$
$$+[2j-c, +]\,(a_{2j-2i-2}\epsilon_i' + a_{2i}\epsilon'_{j-i-1}) + [2j-c, -]\,(a_{-2j-2i-2}\epsilon_i + a_{2i}\epsilon_{-j-i-1})\} = 0.$$

Permissible changes in i make it possible to reduce all the suffixes of ϵ, ϵ' to the form i, and the simpler equations

$$\left.\begin{aligned}
&\sum_i \{[2j+c,\ 2i+c]\,a_{-2j+2i}\epsilon_i + [2j+c,\ 2i+2j]\,a_{2i+2j}\epsilon_i' \\
&\qquad + 2\,[2j+c, +]\,a_{2j-2i-2}\epsilon_i + 2\,[2j+c, -]\,a_{-2j-2i-2}\,\epsilon_i'\} = 0 \\
&\sum_i \{[2j-c,\ 2i-c]\,a_{-2j+2i}\epsilon_i' + [2j-c,\ 2i+2j]\,a_{2i+2j}\epsilon_i \\
&\qquad + 2\,[2j-c, +]\,a_{2j-2i-2}\epsilon_i' + 2\,[2j-c, -]\,a_{-2j-2i-2}\epsilon_i\} = 0
\end{aligned}\right\} \dots (20)$$

are thus obtained. Since the numerical value of m is introduced from the outset and c has been determined, the coefficients of ϵ_i, ϵ_i' are numbers, which in general become rapidly smaller at a distance from the central term. The equations can therefore be solved by continued approximation. As they determine the ratios only of ϵ_i, ϵ_i', it is possible to put

$$\epsilon_0 - \epsilon_0' = 1, \quad \epsilon_i = b_i\epsilon_0 + \beta_i\epsilon_0', \quad \epsilon_i' = b_i'\epsilon_0 + \beta_i'\epsilon_0'.$$

The equations for $j = \pm 1, \pm 2, \ldots$ will then serve to determine the coefficients b_i, β_i, b_i', β_i', where $b_0 = \beta_0' = 1$, $\beta_0 = b_0' = 0$. For $j = 0$,

$$\left.\begin{aligned} 0 = \ldots &+ [\mathrm{c}, 2+\mathrm{c}]\, a_2\epsilon_1 + [\mathrm{c}, 2]\, a_2\epsilon_1' + 2[\mathrm{c}, +]\, a_{-4}\epsilon_1 + 2[\mathrm{c}, -]\, a_{-4}\epsilon_1' \\ &- a_0\epsilon_0 \qquad\qquad\qquad\qquad + 2[\mathrm{c}, +]\, a_{-2}\epsilon_0 + 2[\mathrm{c}, -]\, a_{-2}\epsilon_0' \\ &+ [\mathrm{c}, -2+\mathrm{c}]\, a_{-2}\epsilon_{-1} + [\mathrm{c}, -2]\, a_{-2}\epsilon'_{-1} + 2[\mathrm{c}, +]\, a_0\epsilon_{-1} + 2[\mathrm{c}, -]\, a_0\epsilon'_{-1} + \ldots \end{aligned}\right\} \quad (21)$$

with a similar equation obtained by changing the sign of c and interchanging ϵ. ϵ'. Either of these two equations, with $\epsilon_0 - \epsilon_0' = 1$, determines ϵ_0 and ϵ_0', and hence ϵ_i, ϵ_i' in general. The two must lead to the same result, and together are merely a check on the value of c, which, had it not been determined otherwise, could in theory be deduced from the whole set of these equations.

240. Before continuing the development of a method the whole aim of which is a systematic advance towards great accuracy in the complete results, and which is therefore apt to obscure the main features of the actual motion of the Moon, it will be well to consider the kind of results which have already been obtained implicitly or can be readily deduced. For this purpose a low order of approximation must be adopted and m^4 will be neglected. Then it is easily found that

$$a_2 = [2, +] = \tfrac{3}{16}\mathrm{m}^2 + \tfrac{1}{2}\mathrm{m}^3, \quad a_{-2} = [-2, -] = -\tfrac{19}{16}\mathrm{m}^2 - \tfrac{5}{3}\mathrm{m}^3$$

$$2M_0 = 1 + 2\mathrm{m} + \tfrac{5}{2}\mathrm{m}^2, \quad 2M_1 = 2M_{-1} = \tfrac{3}{2}\mathrm{m}^2 + \tfrac{19}{4}\mathrm{m}^3$$

$$U_0 = 1, \quad U_1 = \tfrac{3}{8}\mathrm{m}^2 + 3\mathrm{m}^3, \quad U_{-1} = -\tfrac{19}{8}\mathrm{m}^2 - \tfrac{10}{3}\mathrm{m}^3$$

$$\Theta_0 = -2M_0 + 2(U_0 + \mathrm{m})^2 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2$$

$$\Theta_1 = -2M_1 + 2(U_0 + \mathrm{m})(U_1 + U_{-1}) - (U_1 - U_{-1}) = -\tfrac{15}{2}\mathrm{m}^2 - \tfrac{57}{4}\mathrm{m}^3.$$

To the order named, the combination of (16) with (17) gives

$$\begin{aligned} \mathrm{c}_0 &= \sqrt{\Theta_0} + \tfrac{1}{4}\Theta_1^2/(1 - \Theta_0)\sqrt{\Theta_0} \\ &= 1 + \mathrm{m} - \tfrac{3}{4}\mathrm{m}^2 - \tfrac{201}{32}\mathrm{m}^3 = 1{\cdot}07263 \end{aligned}$$

and similarly

$$\begin{aligned} \mathrm{g}_0 &= \sqrt{(2M_0)} + M_1^2/(1 - 2M_0)\sqrt{(2M_0)} \\ &= 1 + \mathrm{m} + \tfrac{3}{4}\mathrm{m}^2 - \tfrac{33}{32}\mathrm{m}^3 = 1{\cdot}08521. \end{aligned}$$

The numerical value of g_0, corresponding to $\mathrm{m} = 0{\cdot}08085$, is much nearer the truth than that of c_0. Also it follows from (11) that

$$\mathbf{a} = a\,(1 - \tfrac{1}{6}\mathrm{m}^2 + \tfrac{1}{3}\mathrm{m}^3).$$

Then (12) give

$$r\cos(v - nt - \epsilon) = \mathbf{a}\{1 - (\mathrm{m}^2 + \tfrac{7}{6}\mathrm{m}^3)\cos 2\xi\}$$

$$r\sin(v - nt - \epsilon) = \mathbf{a}(\tfrac{11}{8}\mathrm{m}^2 + \tfrac{13}{6}\mathrm{m}^3)\sin 2\xi$$

whence

$$v = nt + \epsilon + (\tfrac{11}{8}\mathrm{m}^2 + \tfrac{13}{6}\mathrm{m}^3)\sin 2\xi$$

$$r = a\{1 - \tfrac{1}{6}\mathrm{m}^2 + \tfrac{1}{3}\mathrm{m}^3 - (\mathrm{m}^2 + \tfrac{7}{6}\mathrm{m}^3)\cos 2\xi\}.$$

Terms depending on m only are called variational terms. The coefficient of the principal term of the *variation in longitude* is thus

$$\tfrac{11}{8}\mathrm{m}^2 + \tfrac{13}{6}\mathrm{m}^3 = 0{\cdot}01013 = 2090''$$

which is some 16″ in defect of the true value. This term was discovered observationally by Tycho Brahe, and its period, indicated by 2ξ (or $2D$ in Delaunay's notation), is half a synodic month.

241. The equations (20) for $j = \pm 1$, when the leading terms only are retained, become simply

$$\epsilon_1 = \{[2 + \mathrm{c}, \mathrm{c}]\, a_{-2} + 2\,[2 + \mathrm{c}, +]\}\, \epsilon_0 + [2 + \mathrm{c}, 2]\, a_2\, \epsilon_0'$$

$$\epsilon_{-1} = [-2 + \mathrm{c}, \mathrm{c}]\, a_2\, \epsilon_0 + \{[-2 + \mathrm{c}, -2]\, a_{-2} + 2\,[-2 + \mathrm{c}, -]\}\, \epsilon_0'$$

$$\epsilon_1' = [2 - \mathrm{c}, 2]\, a_2\, \epsilon_0 + \{[2 - \mathrm{c}, -\mathrm{c}]\, a_{-2} + 2\,[2 - \mathrm{c}, +]\}\, \epsilon_0'$$

$$\epsilon'_{-1} = \{[-2 - \mathrm{c}, -2]\, a_{-2} + 2\,[-2 - \mathrm{c}, -]\}\, \epsilon_0 + [-2 - \mathrm{c}, -\mathrm{c}]\, a_2 \epsilon_0'.$$

It is to be noticed that $[x, y]$, $[x, \pm]$ contain as a divisor

$$D_x = 2x^2 - 2 - 4\mathrm{m} + \mathrm{m}^2$$

and that this has the factor m when $\pm x = 2 - \mathrm{c}$. It is easily found that

$$[2 + \mathrm{c}, \mathrm{c}] = -\tfrac{7}{24},\quad [2 + \mathrm{c}, 2] = -\tfrac{5}{8},\quad [2 + \mathrm{c}, +] = \tfrac{5}{192}\mathrm{m}^2$$

$$[-2 - \mathrm{c}, -\mathrm{c}] = -\tfrac{1}{8},\quad [-2 - \mathrm{c}, -2] = -\tfrac{11}{24},\quad [-2 - \mathrm{c}, -] = -\tfrac{71}{192}\mathrm{m}^2$$

$$[-2 + \mathrm{c}, \mathrm{c}] = \tfrac{3}{4}\mathrm{m}^{-1} + \tfrac{53}{16},\quad [-2 + \mathrm{c}, -2] = \tfrac{3}{4}\mathrm{m}^{-1} + \tfrac{5}{16}$$

$$[-2 + \mathrm{c}, -] = \tfrac{45}{32}\mathrm{m} + \tfrac{495}{128}\mathrm{m}^2,\quad [2 - \mathrm{c}, +] = -\tfrac{15}{32}\mathrm{m} - \tfrac{261}{128}\mathrm{m}^2$$

$$[2 - \mathrm{c}, 2] = -\tfrac{1}{4}\mathrm{m}^{-1} - \tfrac{55}{16},\quad [2 - \mathrm{c}, -\mathrm{c}] = -\tfrac{1}{4}\mathrm{m}^{-1} - \tfrac{7}{16}$$

as far as the present low order of approximation requires. Hence with the approximate values of a_2, a_{-2},

$$\epsilon_1 = \tfrac{51}{128}\mathrm{m}^2\epsilon_0 - \tfrac{15}{128}\mathrm{m}^2\epsilon_0'$$

$$\epsilon_{-1} = (\tfrac{9}{64}\mathrm{m} + \tfrac{255}{256}\mathrm{m}^2)\,\epsilon_0 + (\tfrac{123}{64}\mathrm{m} + \tfrac{1565}{256}\mathrm{m}^2)\,\epsilon_0'$$

$$\epsilon_1' = -(\tfrac{3}{64}\mathrm{m} + \tfrac{197}{256}\mathrm{m}^2)\,\epsilon_0 - (\tfrac{41}{64}\mathrm{m} + \tfrac{2413}{768}\mathrm{m}^2)\,\epsilon_0'$$

$$\epsilon'_{-1} = -\tfrac{25}{128}\mathrm{m}^2\epsilon_0 - \tfrac{3}{128}\mathrm{m}^2\epsilon_0'.$$

It has been seen how the order of ϵ_{-1}, ϵ_1' is lowered by the divisor D_x. A similar circumstance affects the coefficients of (21) more seriously, since

$$D_\mathrm{c} = 2\mathrm{c}^2 - 2 - 4\mathrm{m} + \mathrm{m}^2 = -\tfrac{225}{8}\mathrm{m}^3.$$

The disappearance of the terms below m^3 explains why an extremely accurate value of c is required in the numerical development. Without continuing the series for c beyond m^3, D_c is here limited to a single term, and therefore only the terms of the very lowest order in (21) can be taken into account. This equation is thus reduced to

$$[\mathrm{c}, 2]\, a_2\epsilon_1' - \epsilon_0 + [\mathrm{c}, -2+\mathrm{c}]\, a_{-2}\epsilon_{-1} + 2\,[\mathrm{c}, +]\, a_0\epsilon_{-1} = 0$$

where

$$[\mathrm{c}, 2] = [\mathrm{c}, -2+\mathrm{c}] = -\tfrac{16}{225}\mathrm{m}^{-3}, \quad [\mathrm{c}, +] = -\tfrac{2}{15}\mathrm{m}^{-1}.$$

Hence

$$\tfrac{1}{75}\left(\tfrac{3}{64}\epsilon_0 + \tfrac{41}{64}\epsilon_0'\right) - \epsilon_0 + \left(\tfrac{19}{225} - \tfrac{4}{15}\right)\left(\tfrac{9}{64}\epsilon_0 + \tfrac{123}{64}\epsilon_0'\right) = 0$$

which gives quite simply $3\epsilon_0 + \epsilon_0' = 0$, and with $\epsilon_0 - \epsilon_0' = 1$, $\epsilon_0 = \frac{1}{4}$, $\epsilon_0' = -\frac{3}{4}$. These values, though representing only the terms of zero order in m, are true within 1 per cent. It follows that

$$\epsilon_1 = \tfrac{3}{16}\mathrm{m}^2, \quad \epsilon_{-1} = -\tfrac{45}{32}\mathrm{m} - \tfrac{555}{128}\mathrm{m}^2$$

$$\epsilon_1' = \tfrac{15}{32}\mathrm{m} + \tfrac{277}{128}\mathrm{m}^2, \quad \epsilon'_{-1} = -\tfrac{1}{32}\mathrm{m}^2$$

where, owing to the imperfect values of ϵ_0, ϵ_0', the second terms in ϵ_{-1}, ϵ_1' may also be defective.

242. The terms thus found in (18) are

$$u = \mathbf{a}\mathrm{e}\zeta\,(\epsilon_0\zeta^{\mathrm{c}} + \epsilon_0'\zeta^{-\mathrm{c}} + \epsilon_1\zeta^{2+\mathrm{c}} + \epsilon_{-1}\zeta^{-2+\mathrm{c}} + \epsilon_1'\zeta^{2-\mathrm{c}} + \epsilon'_{-1}\zeta^{-2-\mathrm{c}})$$

$$s = \mathbf{a}\mathrm{e}\zeta^{-1}\,(\epsilon_0\zeta^{-\mathrm{c}} + \epsilon_0'\zeta^{\mathrm{c}} + \epsilon_1\zeta^{-2-\mathrm{c}} + \epsilon_{-1}\zeta^{2-\mathrm{c}} + \epsilon_1'\zeta^{-2+\mathrm{c}} + \epsilon'_{-1}\zeta^{2+\mathrm{c}})$$

to which correspond (§ 230)

$$r\cos(v - nt - \epsilon) = \mathbf{a}\mathrm{e}\,\{(\epsilon_0 + \epsilon_0')\cos\phi + (\epsilon_1 + \epsilon'_{-1})\cos(2\xi + \phi) + (\epsilon_1' + \epsilon_{-1})\cos(2\xi - \phi)\}$$

$$r\sin(v - nt - \epsilon) = \mathbf{a}\mathrm{e}\,\{(\epsilon_0 - \epsilon_0')\sin\phi + (\epsilon_1 - \epsilon'_{-1})\sin(2\xi + \phi) + (\epsilon_1' - \epsilon_{-1})\sin(2\xi - \phi)\}$$

where

$$\phi = \mathrm{c}\,(n - n')\,(t - t_1)$$

is the argument of the trigonometrical term corresponding to ζ^{c}. These terms are additive to the variational terms already obtained.

The fundamental terms are

$$r\cos(v - nt - \epsilon) = \mathbf{a}\,(1 - \tfrac{1}{2}\mathrm{e}\cos\phi)$$

$$r\sin(v - nt - \epsilon) = \mathbf{a}\mathrm{e}\sin\phi.$$

Now in elliptic motion (24) and (25) of Chapter IV give, to the first order in e,

$$r\cos w = a\,(-\tfrac{3}{2}e + \cos M + \tfrac{1}{2}e\cos 2M)$$

$$r\sin w = a\,(\qquad\quad \sin M + \tfrac{1}{2}e\sin 2M)$$

whence

$$r\cos(w - M) = a\,(1 - e\cos M)$$

$$r\sin(w - M) = 2ae\sin M.$$

These can be identified with the former by putting $\mathbf{a} = a$, $\mathrm{e} = 2e$, $\phi = M$, and

$$\begin{aligned} v &= nt + \epsilon + w - M \\ &= w + \{n - \mathrm{c}(n - n')\} t + \epsilon + \mathrm{c}(n - n') t_1 \\ &= w + \{1 - \mathrm{c}/(1 + \mathrm{m})\} nt + \text{const.} \end{aligned}$$

This shows that to this extent the motion of the Moon is purely elliptic, with eccentricity $\frac{1}{2}\mathrm{e}$, but that this motion is referred to a line rotating uniformly, given by

$$v_0 = \{1 - \mathrm{c}/(1 + \mathrm{m})\} nt = (\tfrac{3}{4}\mathrm{m}^2 + \tfrac{177}{32}\mathrm{m}^3 + \ldots) nt.$$

Thus c determines the motion of the lunar perigee, which completes a revolution in the direct sense in rather less than 9 years. The above approximation gives 128 sidereal months or 3500 days.

In the older lunar theories, beginning with Clairaut, the rotating elliptic orbit is adopted in the first approximation.

243. The result of collecting the terms found so far as necessary is

$$\begin{aligned} r\cos(v - nt - \epsilon) &= \mathbf{a}\,\{1 - \mathrm{m}^2\cos 2\xi - \tfrac{1}{2}\mathrm{e}\cos\phi \\ &\qquad - (\tfrac{15}{16}\mathrm{m} + \tfrac{139}{64}\mathrm{m}^2)\,\mathrm{e}\cos(2\xi - \phi) + \tfrac{5}{32}\mathrm{m}^2\mathrm{e}\cos(2\xi + \phi)\} \\ r\sin(v - nt - \epsilon) &= \mathbf{a}\,\{\tfrac{11}{8}\mathrm{m}^2\sin 2\xi + \mathrm{e}\sin\phi \\ &\qquad + (\tfrac{15}{8}\mathrm{m} + \tfrac{13}{2}\mathrm{m}^2)\,\mathrm{e}\sin(2\xi - \phi) + \tfrac{7}{32}\mathrm{m}^2\mathrm{e}\sin(2\xi + \phi)\}. \end{aligned}$$

The effect of dividing the latter by the former is to add to the second series the terms

$$\mathrm{m}^2\mathrm{e}\,(\cos 2\xi\sin\phi + \tfrac{11}{16}\sin 2\xi\cos\phi) = \mathrm{m}^2\mathrm{e}\,\{\tfrac{27}{32}\sin(2\xi + \phi) - \tfrac{5}{32}\sin(2\xi - \phi)\}.$$

Hence the longitude is approximately

$$\begin{aligned} v = nt + \epsilon + \tfrac{11}{8}\mathrm{m}^2\sin 2\xi + \mathrm{e}\sin\phi \\ + (\tfrac{15}{8}\mathrm{m} + \tfrac{203}{32}\mathrm{m}^2)\,\mathrm{e}\sin(2\xi - \phi) + \tfrac{17}{16}\mathrm{m}^2\mathrm{e}\sin(2\xi + \phi). \end{aligned}$$

As a constant of integration introduced at one stage of the present method, e may be defined in any suitable way for the later stages. Its value depends on the exact definition adopted and will be found by comparing the final results with observation. Thus $\frac{1}{2}\mathrm{e}$ as defined by Brown is not to be identified with the e of Delaunay, for example. The difference is not great, however, and its value may be taken to be 0·0549. Thus the coefficient of the *principal elliptic term in longitude*, $\mathrm{e}\sin\phi$, is of the order 6°·3.

The term next in importance has the argument $2\xi - \phi$ (or $2D - l$ in Delaunay's notation). The coefficient is right to the order given, though the above derivation left this doubtful, and its value gives

$$(\tfrac{15}{8}\mathrm{m} + \tfrac{203}{32}\mathrm{m}^2)\,\mathrm{e} = 73' \text{ nearly.}$$

The true coefficient, depending on e alone, is 4608″. This inequality is the largest true perturbation in the Moon's motion and is known as the *Evection*. Its discovery from observation is due to Ptolemy.

The term with the argument $2\xi + \phi$ (or $2D + l$) is much smaller. The above coefficient gives 157″, while the true value is about 175″ for the part depending on e alone. It will be noticed that the greater part of it is due not to a true perturbation in the rectangular coordinates but to interference between the variation and the principal elliptic term in deriving the longitude.

244. The terms depending on the first power of the solar eccentricity e' will be next considered. With $z = 0$ and the solar parallax still neglected, $\Omega = \Omega_2$ and (4), (5) become

$$D^2(us) - Du\,.\,Ds + 2\mathrm{m}(sDu - uDs) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C - 3\Omega_2 + D^{-1}(D_t\Omega_2)$$

$$D(uDs - sDu - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) \qquad = s\frac{\partial\Omega_2}{\partial s} - u\frac{\partial\Omega_2}{\partial u}$$

where (3) gives

$$\Omega_2 = \mathrm{m}^2\frac{a'^3}{r_1^3}(3r^2S^2 - r^2) - \tfrac{1}{4}\mathrm{m}^2\{3(u+s)^2 - 4us\}.$$

Now

$$rS = (XX' + YY')\,r_1^{-1} = \tfrac{1}{2}(u+s)\cos\chi' - \tfrac{1}{2}\iota(u-s)\sin\chi'$$

where (§ 223) $\chi' = v' - n't - \epsilon' = v' - \phi'$ is the solar equation of the centre. Hence

$$r^2S^2 = \tfrac{1}{4}(u^2+s^2)\cos 2\chi' + \tfrac{1}{2}us - \tfrac{1}{4}\iota(u^2-s^2)\sin 2\chi'$$

and therefore

$$\Omega_2 = \mathrm{m}^2\frac{a'^3}{r_1^3}\{\tfrac{3}{4}(u^2+s^2)\cos 2\chi' + \tfrac{1}{2}us - \tfrac{3}{4}\iota(u^2-s^2)\sin 2\chi'\} - \tfrac{1}{4}\mathrm{m}^2(3u^2 + 3s^2 + 2us)$$

where u, s have the values given by the variational curve. The Sun's mean anomaly is

$$\phi' = n'(t - t_3) = \mathrm{m}(n - n')(t - t_3) = -\iota\log\zeta_3^{\mathrm{m}}.$$

The whole disturbing function must ultimately be developed in powers of ζ_3^{m} as far as necessary, the coefficients involving u, s, a'^{-1} and e'. But for the immediate purpose it is easily verified that to the first order in e',

$$\frac{a'^3}{r_1^3} = \frac{a'^3}{r_1^3}\cos 2\chi' = 1 + 3e'\cos\phi', \quad \frac{a'^3}{r_1^3}\sin 2\chi' = 4e'\sin\phi'.$$

Hence

$$\Omega_2 = \tfrac{3}{4}\mathrm{m}^2e'\{u^2(-\tfrac{1}{2}\zeta_3^{\mathrm{m}} + \tfrac{7}{2}\zeta_3^{-\mathrm{m}}) + s^2(\tfrac{7}{2}\zeta_3^{\mathrm{m}} - \tfrac{1}{2}\zeta_3^{-\mathrm{m}}) + us(\zeta_3^{\mathrm{m}} + \zeta_3^{-\mathrm{m}})\}$$

$$D_t\Omega_2 = \tfrac{3}{4}\mathrm{m}^3e'\{u^2(-\tfrac{1}{2}\zeta_3^{\mathrm{m}} - \tfrac{7}{2}\zeta_3^{-\mathrm{m}}) + s^2(\tfrac{7}{2}\zeta_3^{\mathrm{m}} + \tfrac{1}{2}\zeta_3^{-\mathrm{m}}) + us(\zeta_3^{\mathrm{m}} - \zeta_3^{-\mathrm{m}})\}.$$

Thus the right-hand members of the equations at the beginning of this section will be of the form

$$\mathbf{a}^2e'\Sigma E_{2i+p\mathrm{m}}\zeta^{2i+p\mathrm{m}}, \quad \mathbf{a}^2e'\Sigma E'_{2i+p\mathrm{m}}\zeta^{2i+p\mathrm{m}}$$

for, as in § 238, the suffix of ζ_3 may be suppressed in the calculation with the proper understanding as to the argument corresponding to ζ^{m} in the results. The solution is of the form

$$u = \mathbf{a}\zeta\sum_i\sum_p A_{2i+p\mathrm{m}}\zeta^{2i+p\mathrm{m}}, \quad s = \mathbf{a}\zeta^{-1}\sum_i\sum_p A_{-2i-p\mathrm{m}}\zeta^{2i+p\mathrm{m}}$$

where

$$A_{2i} = a_{2i}, \quad A_{2i+\mathrm{m}} = e'\eta_i, \quad A_{2i-\mathrm{m}} = e'\eta_i'$$

and p has the values 0, ± 1 only, until higher powers of e' are taken into account. The solution follows the same course as in § 239 except that there are now terms on the right-hand side of the equations. The equations of condition corresponding to (20) are thus

$$\sum_i \{[2j+\mathrm{m}, 2i+\mathrm{m}]\, a_{-2j+2i}\,\eta_i + [2j+\mathrm{m}, 2i+2j]\, a_{2i+2j}\,\eta_i'$$
$$+ 2\,[2j+\mathrm{m}, +]\, a_{2j-2i-2}\,\eta_i + 2\,[2j+\mathrm{m}, -]\, a_{-2j-2i-2}\,\eta_i'\} = E''_{2j+\mathrm{m}}.$$

This form results from the linear combination of a pair of equations obtained by comparing coefficients of $\zeta^{2j+\mathrm{m}}$ and in these the leading terms by analogy with (9) are respectively

$$\ldots + \{4j'^2 + 2j' + 1 + 4\mathrm{m}\,(j'+1) + \tfrac{9}{2}\mathrm{m}^2\}\, a_0 e'\eta_j$$
$$+ \{4j'^2 - 2j' + 1 - 4\mathrm{m}\,(j'-1) + \tfrac{9}{2}\mathrm{m}^2\}\, a_0 e'\eta'_{-j} + \ldots = e'E_{2j+\mathrm{m}}$$
$$\ldots - 4j'\,(1 + j' + \mathrm{m}\}\, a_0 e'\eta_j - 4j'\,(1 - j' + \mathrm{m})\, a_0 e'\eta'_{-j} + \ldots = e'E'_{2j+\mathrm{m}}$$

where j' is written for $j + \frac{1}{2}\mathrm{m}$. The combination is such that the coefficient of η'_{-j} vanishes and that of η_j becomes -1. Hence

$$E''_{2j+\mathrm{m}} = \frac{4j'\,(1 - j' + \mathrm{m})\, E_{2j+\mathrm{m}} + \{4j'^2 - 2j' + 1 - 4\mathrm{m}\,(j'-1) + \tfrac{9}{2}\mathrm{m}^2\}\, E'_{2j+\mathrm{m}}}{4j'^2\,(8j'^2 - 2 - 4\mathrm{m} + \mathrm{m}^2)}.$$

The divisor, which appears also in the symbolical coefficients [ ], becomes small only through the factor j', when $j = 0$, $4j'^2 = \mathrm{m}^2$.

245. The calculation of η_j, η_j' when m is given its numerical value at the outset, proceeds as in the case of ϵ_j, ϵ_j' with this difference, that the equations contain definite right-hand members. A particular solution of the differential equations is required, representing a forced disturbance of the steady variational motion. Hence no new constant of integration enters.

The machinery is of course absurdly elaborate when only the main parts of the leading terms are sought, but this plan will be pursued. It is easily found that

$$\Omega_2 = \tfrac{3}{4}\mathrm{m}^2 e'\mathbf{a}^2\,\{-\tfrac{1}{2}\,(\zeta^{2+\mathrm{m}} + \zeta^{-2-\mathrm{m}}) + \tfrac{7}{2}\,(\zeta^{2-\mathrm{m}} + \zeta^{-2+\mathrm{m}}) + (1 + 6a_{-2})\,(\zeta^{\mathrm{m}} + \zeta^{-\mathrm{m}})\}$$

with the neglect of m in the coefficients of $\zeta^{\pm 2 \pm \mathrm{m}}$, but not $\zeta^{\pm\mathrm{m}}$. The operator D_t applies to $\zeta^{\pm\mathrm{m}}$ only and gives a multiplier $\pm$ m to every term, while the operator D^{-1} applies to ζ generally and gives divisors $\pm 2 \pm \mathrm{m}$ or $\pm$ m. Hence to the same order in m

$$D^{-1}\,(D_t\Omega_2) = \tfrac{3}{4}\mathrm{m}^2 e'\mathbf{a}^2\,\{(1 + 6a_{-2})\,(\zeta^{\mathrm{m}} + \zeta^{-\mathrm{m}})\}.$$

Also

$$s\frac{\partial\Omega_2}{\partial s} - u\frac{\partial\Omega_2}{\partial u} = \tfrac{3}{2}\mathrm{m}^2 e'\mathbf{a}^2\,\{\tfrac{1}{2}\,(\zeta^{2+\mathrm{m}} - \zeta^{-2-\mathrm{m}}) - \tfrac{7}{2}\,(\zeta^{2-\mathrm{m}} - \zeta^{-2+\mathrm{m}}) + 8a_{-2}\,(\zeta^{\mathrm{m}} - \zeta^{-\mathrm{m}})\}$$

Hence

$$E_{\mathrm{m}} = E_{-\mathrm{m}} = -\tfrac{3}{2}\mathrm{m}^2(1 + 6a_{-2}), \quad E_{\mathrm{m}}' = -E'_{-\mathrm{m}} = 12\mathrm{m}^2 a_{-2}$$
$$E_{\mathrm{m}}'' = (-\mathrm{m}^{-1} + \tfrac{3}{2})\,E_{\mathrm{m}} - \tfrac{1}{2}\mathrm{m}^{-2}E_{\mathrm{m}}' = \tfrac{3}{2}\mathrm{m} + \tfrac{39}{8}\mathrm{m}^2$$
$$E''_{-\mathrm{m}} = (\mathrm{m}^{-1} - \tfrac{1}{2})\,E_{-\mathrm{m}} - \tfrac{1}{2}\mathrm{m}^{-2}E'_{-\mathrm{m}} = -\tfrac{3}{2}\mathrm{m} - \tfrac{51}{8}\mathrm{m}^2.$$

Thus η_0, η_0' must be of the first order in m and give rise to terms of at least the third order in the equations for $j = \pm 1$. These contain no small divisor and for the lowest order they give immediately:

$$-\eta_1 = E''_{2+\mathrm{m}} = \tfrac{1}{8}E'_{2+\mathrm{m}} = \tfrac{3}{32}\,\mathrm{m}^2$$
$$-\eta_1' = E''_{2-\mathrm{m}} = \tfrac{1}{8}E'_{2-\mathrm{m}} = -\tfrac{21}{32}\,\mathrm{m}^2$$
$$-\eta_{-1} = E''_{-2+\mathrm{m}} = -\tfrac{1}{3}E_{-2+\mathrm{m}} + \tfrac{7}{24}E'_{-2+\mathrm{m}} = \tfrac{133}{32}\,\mathrm{m}^2$$
$$-\eta'_{-1} = E''_{-2-\mathrm{m}} = -\tfrac{1}{3}E_{-2-\mathrm{m}} + \tfrac{7}{24}E'_{-2-\mathrm{m}} = -\tfrac{19}{32}\mathrm{m}^2.$$

Coefficients of the form $[\mathrm{m}, y]$ are of the order -1 in m, but they multiply terms of at least the fourth order in the equations for $j = 0$. These give therefore to the second order

$$-\eta_0 \quad + 2\,[\mathrm{m}, +]\,a_0\eta_{-1} \quad + 2\,[\mathrm{m}, -]\,a_0\eta'_{-1} = E''_{\mathrm{m}}$$
$$-\eta_0' + 2\,[-\mathrm{m}, +]\,a_0\eta'_{-1} + 2\,[-\mathrm{m}, -]\,a_0\eta_{-1} = E''_{-\mathrm{m}}$$

where

$$[\mathrm{m}, +] = [-\mathrm{m}, +] = -\tfrac{3}{4}, \quad [\mathrm{m}, -] = [-\mathrm{m}, -] = \tfrac{3}{4}.$$

Accordingly

$$-\eta_0 = \tfrac{3}{2}\mathrm{m} - \tfrac{9}{4}\mathrm{m}^2, \quad -\eta_0' = -\tfrac{3}{2}\mathrm{m} + \tfrac{3}{4}\mathrm{m}^2.$$

Thus the principal terms depending on the solar eccentricity may be put in the form

$$r\cos(v - nt - \epsilon)$$
$$= \mathbf{a}e'\,\{(\eta_0 + \eta_0')\cos\phi' + (\eta_1 + \eta'_{-1})\cos(2\xi + \phi') + (\eta_1' + \eta_{-1})\cos(2\xi - \phi')\}$$
$$= \mathbf{a}e'\,\{\tfrac{3}{2}\mathrm{m}^2\cos\phi' + \tfrac{1}{2}\mathrm{m}^2\cos(2\xi + \phi') - \tfrac{7}{2}\mathrm{m}^2\cos(2\xi - \phi')\}$$

$$r\sin(v - nt - \epsilon)$$
$$= \mathbf{a}e'\,\{(\eta_0 - \eta_0')\sin\phi' + (\eta_1 - \eta'_{-1})\sin(2\xi + \phi') + (\eta_1' - \eta_{-1})\sin(2\xi - \phi')\}$$
$$= \mathbf{a}e'\,\{-3\,(\mathrm{m} - \mathrm{m}^2)\sin\phi' - \tfrac{11}{16}\mathrm{m}^2\sin(2\xi + \phi') + \tfrac{77}{16}\mathrm{m}^2\sin(2\xi - \phi')\}.$$

In deriving the longitude there are no interfering terms of this order, and the last line without **a** gives the additional terms depending on e'. The term with argument ϕ' (or l') is called the *Annual Equation* after its period. The value of e' is 0·01675 and the coefficient of this part of the term, $-3e'(\mathrm{m} - \mathrm{m}^2)$, is $-770''$ as compared with the complete value $-659''$. For the argument $2\xi - \phi'$ (or $2D - l'$) the coefficient $\tfrac{77}{16}e'\mathrm{m}^2$ is $+109''$, the true value being $+152''$, and for the argument $2\xi + \phi'$ (or $2D + l'$) the coefficient $-\tfrac{11}{16}e'\mathrm{m}^2$ is $-15''\!\cdot\!5$, the true value being $-21''\!\cdot\!6$. The discrepancies are considerable and show that the parts depending on higher powers of m are large. As series in m the coefficients converge slowly, and hence the great

advantage of the Hill-Brown method, which by employing an accurate *numerical* value of m from the beginning avoids expansions in this parameter altogether.

246. In deriving the terms with the characteristic a'^{-1} alone, e' is neglected and therefore $\Omega_2 = 0$, $D_t\Omega = 0$, and

$$\Omega = \Omega_3 = 2\mathrm{m}^2 a'^{-1} P_3 r^3 = \mathrm{m}^2 a'^{-1} (5r^3S^3 - 3r^3S)$$
$$= \tfrac{1}{8}\mathrm{m}^2 a'^{-1} \{5(u+s)^3 - 12us(u+s)\}$$

since $rS = X = \frac{1}{2}(u+s)$ when $e' = 0$. The terms on the right-hand side of (4), (5) are thus

$$-4\Omega_3 = -\tfrac{1}{2}\mathrm{m}^2 a'^{-1}\{5(u^3+s^3) + 3us(u+s)\} = \mathbf{a}^3 a'^{-1} \Sigma E_{2i+1} \zeta^{2i+1}$$

$$s\frac{\partial\Omega_3}{\partial s} - u\frac{\partial\Omega_3}{\partial u} = -\tfrac{3}{8}\mathrm{m}^2 a'^{-1}\{5(u^3-s^3) + us(u-s)\} = \mathbf{a}^3 a'^{-1} \Sigma E'_{2i+1} \zeta^{2i+1}$$

respectively. The additional terms required in the solution must be of the form

$$u = \mathbf{a}^2 a'^{-1} \zeta \Sigma \alpha_{2i+1} \zeta^{2i+1}, \quad s = \mathbf{a}^2 a'^{-1} \zeta^{-1} \Sigma \alpha_{-2i-1} \zeta^{2i+1}$$

in order to produce odd powers of ζ. Similarly Ω_4 has the factor a'^{-2} and gives rise to terms with the same arguments as the variational terms. The solution follows the same course as for the terms with characteristic e', and the relation connecting E''_{2j+1} with E_{2j+1}, E'_{2j+1} is the same as before when $j' = j + \frac{1}{2}$.

The principal terms are given by $2j + 1 = \pm 1, \pm 3$. The divisor $D_{2j'}$ is of the order m when $j' = \pm\frac{1}{2}$ only. But Ω_3 contains m^2 as a factor. Hence, when terms of the order m^3 are neglected in E'_{2j+1}, m^2 can be neglected in $\mathrm{m}^{-2}\Omega_3$ and the variational coefficients a_2, a_{-2} are not required. Thus it is enough to write

$$-4\Omega_3 = -\tfrac{1}{2}\mathrm{m}^2\mathbf{a}^3 a'^{-1}\{5(\zeta^3+\zeta^{-3}) + 3(\zeta+\zeta^{-1})\}$$

$$s\frac{\partial\Omega_3}{\partial s} - u\frac{\partial\Omega_3}{\partial u} = -\tfrac{3}{8}\mathrm{m}^2\mathbf{a}^3 a'^{-1}\{5(\zeta^3-\zeta^{-3}) + (\zeta-\zeta^{-1})\}$$

and therefore

$$-\alpha_3 = E_3'' = -\tfrac{1}{48}E_3 + \tfrac{7}{144}E_3' = -\tfrac{5}{128}\mathrm{m}^2$$
$$-\alpha_{-3} = E''_{-3} = -\tfrac{5}{48}E_{-3} + \tfrac{13}{144}E'_{-3} = \tfrac{55}{128}\mathrm{m}^2.$$

Also, to the same order in m,

$$E_1'' = (-\tfrac{1}{4}\mathrm{m}^{-1} - \tfrac{9}{16})E_1 + (-\tfrac{1}{4}\mathrm{m}^{-1} - \tfrac{9}{16})E_1' = \tfrac{15}{32}\mathrm{m} + \tfrac{135}{128}\mathrm{m}^2$$
$$E''_{-1} = (\tfrac{3}{4}\mathrm{m}^{-1} + \tfrac{11}{16})E_{-1} + (-\tfrac{3}{4}\mathrm{m}^{-1} - \tfrac{27}{16})E'_{-1} = -\tfrac{45}{32}\mathrm{m} - \tfrac{213}{128}\mathrm{m}^2.$$

The equations for α_1, α_{-1} can be adapted from (21) and its correlative by putting $c = 1$, $\epsilon_0 = \epsilon_1' = \alpha_1$ and $\epsilon_0' = \epsilon_{-1} = \alpha_{-1}$. To the second order in m these give

$$[1, 2]\, a_2\alpha_1 - \alpha_1 + [1, -1]\, a_{-2}\alpha_{-1} + 2[1, +]\, a_0\alpha_{-1} = E_1''$$
$$[-1, 1]\, a_2\alpha_1 - \alpha_{-1} + [-1, -2]\, a_{-2}\alpha_{-1} + 2[-1, -]\, a_0\alpha_{-1} = E''_{-1}$$

whence

$$-\tfrac{3}{32}\mathrm{m}\alpha_1 - \alpha_1 + \tfrac{19}{32}\mathrm{m}\alpha_{-1} - \tfrac{15}{8}\mathrm{m}\alpha_{-1} = \tfrac{15}{32}\mathrm{m} + \tfrac{135}{128}\mathrm{m}^2$$

$$\tfrac{9}{32}\mathrm{m}\alpha_1 - \alpha_{-1} - \tfrac{57}{32}\mathrm{m}\alpha_{-1} + \tfrac{45}{8}\mathrm{m}\alpha_{-1} = -\tfrac{45}{32}\mathrm{m} - \tfrac{213}{128}\mathrm{m}^2$$

and therefore

$$-\alpha_1 = \tfrac{15}{32}\mathrm{m} + \tfrac{45}{16}\mathrm{m}^2, \quad -\alpha_{-1} = -\tfrac{45}{32}\mathrm{m} - \tfrac{111}{16}\mathrm{m}^2.$$

The additional terms in their elementary form are thus

$$r\cos(v - nt - \epsilon) = \mathbf{a}^2 a'^{-1}\{(\alpha_1 + \alpha_{-1})\cos\xi + (\alpha_3 + \alpha_{-3})\cos 3\xi\}$$
$$= \mathbf{a}^2 a'^{-1}\{(\tfrac{15}{16}\mathrm{m} + \tfrac{33}{8}\mathrm{m}^2)\cos\xi - \tfrac{25}{64}\mathrm{m}^2\cos 3\xi\}$$
$$r\sin(v - nt - \epsilon) = \mathbf{a}^2 a'^{-1}\{(\alpha_1 - \alpha_{-1})\sin\xi + (\alpha_3 - \alpha_{-3})\sin 3\xi\}$$
$$= \mathbf{a}^2 a'^{-1}\{-(\tfrac{15}{8}\mathrm{m} + \tfrac{39}{4}\mathrm{m}^2)\sin\xi + \tfrac{15}{32}\mathrm{m}^2\sin 3\xi\}$$

and the last line, divided by **a**, gives the corresponding terms in longitude. The mean parallax of the Sun is $8''\cdot 80$ and of the Moon $3422''\cdot 7$; to the above order $\mathbf{a}/a' = 0\cdot 002571$. This gives $-114''$ for the coefficient of the first term (argument ξ or D) and $1''\cdot 6$ for the coefficient of the second (argument 3ξ or $3D$), whereas the complete values, with the characteristic $\mathbf{a}/a'$ alone, are $-125''$ and under $1''$. The term with argument D is known as the *Parallactic Inequality*. Its period is one *lunation* (or synodic month) and the comparison of its theoretical coefficient with observation gave probably the best determination of the solar parallax until the direct geometrical method based on the observation of minor planets was adopted. This use of the parallactic inequality is not entirely free from objection because the Moon cannot be observed throughout a complete lunation and systematic error may be suspected, due to the varying illumination of the lunar disc.

247. Hitherto the terms of u, s which are of the first order in the characteristics $\mathrm{e}, e', \mathbf{a}a'^{-1}$ have alone been considered. If the third coördinate z be assumed to be of the first order the first two equations of (2) show that u, s contain in addition only terms of the second and higher orders. The third equation of (2) has already been considered in § 236, and when Ω is neglected terms in z of the first order are given by the equation

$$D^2 z = (2\Sigma M_i \zeta^{2i})\, z.$$

Let

$$\eta = \mathrm{g}(n - n')(t - t_2) = -\iota \log \zeta_2^{\mathrm{g}}.$$

Then the general solution is of the form

$$\iota z = \mathbf{a}\mathrm{k}\Sigma k_i(\zeta^{2i+\mathrm{g}} - \zeta^{-2i-\mathrm{g}})$$

where a preliminary value of g has been found in § 240 and k, t_2 represent the two necessary arbitrary constants. As before the suffix of ζ_2 has been suppressed because it does not affect the calculation, though the proper

argument must be retained in the results. The coefficients k_i are determined by equating terms in ζ^{2j+g}, so that

$$k_j(2j+g)^2 = \Sigma 2M_i k_{j-i}$$

and it is possible to write $k_0 = 1$.

In obtaining k_1, k_{-1} to m^2 only it is possible to neglect k_2, k_{-2} and approximate values of M_0, $M_1 = M_{-1}$ have been found in § 240. Thus the equations are

$$(2+g)^2 k_1 \;\;= 2M_0 k_1 \;\;+ 2M_1 k_0$$
$$(2-g)^2 k_{-1} = 2M_0 k_{-1} + 2M_{-1} k_0$$

where

$$(2+g)^2 - 2M_0 = 8,\quad (2-g)^2 - 2M_0 = -4m - 3m^2,\quad 2M_1 = 2M_{-1} = \tfrac{3}{2}m^2 + \tfrac{19}{4}m^3.$$

Hence

$$k_1 = \tfrac{3}{16}m^2,\quad k_{-1} = -\tfrac{3}{8}m - \tfrac{29}{32}m^2$$

and to this order in m

$$\iota z = \;\mathbf{a}k\,\{\zeta^{g} - \zeta^{-g} - (\tfrac{3}{8}m + \tfrac{29}{32}m^2)(\zeta^{-2+g} - \zeta^{2-g}) + \tfrac{3}{16}m^2(\zeta^{2+g} - \zeta^{-2-g})\}$$
$$z = 2\mathbf{a}k\,\{\sin\eta + (\tfrac{3}{8}m + \tfrac{29}{32}m^2)\sin(2\xi - \eta) + \tfrac{3}{16}m^2\sin(2\xi + \eta)\}.$$

248. Here the fundamental term is

$$z = 2\mathbf{a}k\sin\eta = 2\mathbf{a}k\sin\{g(n-n')(t-t_2)\}$$

and its general meaning is easily seen, though the exact definition of k must be adapted to the final approximation and then determined (like e) by direct comparison with observation. The maximum value of z is $2\mathbf{a}k$. But it is also approximately $\mathbf{a}\tan I$, $\mathbf{a}$ being the mean distance in the orbit projected on the plane of the ecliptic and I being the inclination of the orbit to this plane. Hence k is nearly $\frac{1}{2}\tan I$, and differs little from Delaunay's $\gamma = \sin\frac{1}{2}I$. Its provisional value may be taken to be $0{\cdot}0448866 = 9260''$.

At a node $z=0$ and the period between successive returns to the same node is $2\pi/g(n-n')$. In this time the mean motion in longitude is $2\pi n/g(n-n')$. Hence the mean rate of change in the position of the node is

$$\{2\pi n/g(n-n') - 2\pi\} \div 2\pi/g(n-n') = n - g(n-n')$$
$$= n\{1 - g/(1+m)\} = n(-\tfrac{3}{4}m^2 + \tfrac{57}{32}m^3)$$

with the approximate value of g found in § 240. Since this expression is negative the lunar node has a retrogade motion and completes a circuit in 6890 days or 18·9 years, which is reduced by about 100 days when the complete value of g is used. These facts have an important bearing on the theory of eclipse cycles.

In deriving the elementary terms in latitude with the characteristic k it is enough to take from the variational solution

$$r = \mathbf{a}(1 - m^2\cos 2\xi)$$

and to the order m^2 the latitude is

$$z/r = 2k\,\{\sin\eta + (\tfrac{3}{8}m + \tfrac{13}{32}m^2)\sin(2\xi - \eta) + \tfrac{11}{16}m^2\sin(2\xi + \eta)\}.$$

The first term, with argument η (or F in Delaunay's notation) is the principal term in latitude. Its coefficient is $5°8'$. The second term, with argument $2\xi - \eta$ (or $2D - F$), has been called the evection in latitude. Its coefficient as found above is $610''\cdot 6$, the true value being $618''\cdot 4$. The third term, with argument $2\xi + \eta$ (or $2D + F$) has the coefficient $83''\cdot 2$ as compared with the true value $94''\cdot 5$.

249. It is now possible to sketch the whole method of the subsequent development. The greater part of the practical work of calculation has been based not on the homogeneous equations used above, which present advantages in special cases (especially the calculation of long-period terms), but on the original equations (2),

$$D^2u + 2\mathrm{m}Du + \tfrac{3}{2}\mathrm{m}^2(u+s) - \frac{\kappa u}{r^3} = -\frac{\partial\Omega}{\partial s}$$

$$D^2z - \mathrm{m}^2 z - \frac{\kappa z}{r^3} = -\tfrac{1}{2}\frac{\partial\Omega}{\partial z}.$$

It is unnecessary to use the equation in s because $s = f(\zeta^{-1})$ if $u = f(\zeta)$; two real equations are replaced by a single complex one. Also the characteristics entering into u and z are distinct. Hence the treatment of the equations in u and z is also distinct. The order of a characteristic is the sum of the positive powers of the parameters e, e', $\mathrm{a}a'^{-1}$, k which compose it; m is a mere number for this purpose, and retains its identity only in the arguments. Now suppose that a complete solution $u = u_1$, $s = s_1$, $z = z_1$ to the order μ in the characteristics has been obtained. The next step is to find the solution $u = u_1 + u_2$, $s = s_1 + s_2$, $z = z_1 + z_2$, where u_2, s_2, z_2 represent the terms of order $\mu + 1$. Insert these values in the equations, retaining only the first powers of u_2, s_2, z_2. The result is, since $r^2 = us + z^2$,

$$(D+\mathrm{m})^2(u_1+u_2) + \tfrac{1}{2}\mathrm{m}^2(u_1+u_2+3s_1+3s_2) - \kappa(u_1+u_2)r_1^{-3}$$
$$+ \tfrac{3}{2}\kappa u_1 r_1^{-5}(u_1 s_2 + u_2 s_1 + 2z_1z_2) = -\frac{\partial\Omega}{\partial s}$$

$$(D^2 - \mathrm{m}^2)(z_1+z_2) - \kappa(z_1+z_2)r_1^{-3} + \tfrac{3}{2}\kappa z_1 r_1^{-5}(u_1s_2 + u_2s_1 + 2z_1z_2) = -\tfrac{1}{2}\frac{\partial\Omega}{\partial z}.$$

Now terms of order less than $\mu + 1$ must be satisfied identically and therefore terms linear in u_1, s_1, z_1 may be omitted. Also terms of order higher than $\mu + 1$ can be neglected. Hence u_1, s_1, z_1 may be used in calculating Ω, and in conjunction with u_2, s_2, z_2 it is possible to write $u_1 = u_0$, $s_1 = s_0$, $z_1 = 0$, $r_1^2 = u_0 s_0 = \rho_0^2$, where u_0, s_0, $z = 0$ is the variational solution of zero order. Hence the equations reduce to

$$\left.\begin{aligned} &(D+\mathrm{m})^2u_2 + u_2(\tfrac{1}{2}\mathrm{m}^2 + \tfrac{1}{2}\kappa\rho_0^{-3}) + s_2(\tfrac{3}{2}\mathrm{m}^2 + \tfrac{3}{2}\kappa u_0^2\rho_0^{-5}) \\ &\qquad = -\left(\frac{\partial\Omega}{\partial s}\right)_1 + \kappa u_1 r_1^{-3} - (D^2 + 2\mathrm{m}D)u_1 \\ &D^2z_2 - z_2(\mathrm{m}^2 + \kappa\rho_0^{-3}) = -\tfrac{1}{2}\left(\frac{\partial\Omega}{\partial z}\right)_1 + \kappa z_1 r_1^{-3} - D^2 z_1 \end{aligned}\right\} \quad (22)$$

where the terms with D have been retained on the right-hand side, though apparently of order not higher than μ, for a reason to be explained later. For the moment they can be left out of sight.

250. Since the treatment of the two equations is separate but quite similar it will be enough to consider the first. It is convenient to write $u_1 = u_0 + u_1'$, $s_1 = s_0 + s_1'$ and to expand the term $\kappa u_1 r_1^{-3}$ in terms of u_1', s_1', z_1, rejecting the variational part $\kappa u_0 \rho_0^{-3}$ and the linear terms. The form of the known solution has been made sufficiently obvious, and it is clear that the right-hand side, when developed, will contain an aggregate of characteristics λ each of order $\mu + 1$ and each associated with one or more series. Each constituent part may be taken to be of the form

$$A = \mathbf{a}\lambda\zeta\sum_i (A_i\zeta^{2i+\tau} + A'_{-i}\zeta^{-2i-\tau})$$

where

$$\tau = q_1\mathrm{c} + q_2\mathrm{m} + q_3\mathrm{g}$$

q_1, q_2, q_3 having fixed integral values (positive or negative) in the series considered, while $2i$ may have *odd* integral values when $\mathbf{a}a'^{-1}$ occurs in λ.

The part of the solution required to satisfy this series is of the same form

$$u_2 = \mathbf{a}\lambda\zeta\sum_i (\lambda_i\zeta^{2i+\tau} + \lambda'_{-i}\zeta^{-2i-\tau})$$

and λ_i, λ_i' are to be found by inserting this expression in the equation. This may be written

$$(D + \mathrm{m})^2 u_2 + Mu_2 + Ns_2\zeta^2 = A$$

where

$$M = \tfrac{1}{2}\mathrm{m}^2 + \tfrac{1}{2}\kappa\rho_0^{-3} = \Sigma M_i\zeta^{2i}, \quad N\zeta^2 = \tfrac{3}{2}\mathrm{m}^2 + \tfrac{3}{2}\kappa u_0^2\rho_0^{-5} = \zeta^2\Sigma N_i\zeta^{2i}.$$

The series M, in which $M_i = M_{-i}$, has already occurred in the determination of c_0 and g_0. After substitution of the series for u_2, s_2 comparison of the terms in $\zeta^{\pm(2j+\tau)+1}$ on both sides of the equation gives

$$\left.\begin{aligned} (2j + \tau + 1 + \mathrm{m})^2\lambda_j + \sum_i M_i\lambda_{j-i} + \sum_i N_i\lambda'_{i-j} &= A_j \\ (2j + \tau - 1 - \mathrm{m})^2\lambda'_{-j} + \sum_i M_i\lambda'_{-j-i} + \sum_i N_i\lambda_{j+i} &= A'_{-j} \end{aligned}\right\} \quad \text{......(23)}$$

This series of linear equations, in which the coefficients M_i, N_i rapidly diminish, must then be solved by successive approximation. When this has been carried out for each series A and every characteristic λ, all the terms of order $\mu + 1$ in u, s have been determined. The treatment of z is precisely similar.

251. But one important question clearly arises. Is the set of linear equations consistent and definite? If the modulus of the set, which can be written as a symmetrical determinant of infinite order since $M_i = M_{-i}$, $N_i = N_{-i}$, is not zero, the solution is certainly definite. This is the general case. But consider the determination of ϵ_i, ϵ_i' the co-factors of the characteristic e of the first order. By the above method these will be obtained from (23) by putting $A_j = A'_{-j} = 0$ and $\tau = \mathrm{c}$. The consistency of the equations

now requires the modulus to vanish. It is obvious that this condition in fact must lead to a determination of τ which will be identical with the value of c_0, though the latter was found above in a formally different way. When the equations have thus been made consistent the solution only becomes definite when the arbitrary condition $\epsilon_0 - \epsilon_0' = 1$ is added, and this condition is equivalent to a definition of e.

It is now evident that the modulus vanishes whenever $\tau = c$, or for every series based on the same argument as that of the principal elliptic term. The consistency of the linear equations requires a relation between the coefficients A_j, A_j' which may be expressed by equating the modulus to zero after replacing any column in it by the series A_j, A_j'. But owing to the symmetry of the modulus this relation is capable of a much simpler form. Let the equations (23) be multiplied by ϵ_j, ϵ'_{-j} and let the sum be taken for all values of j. Then the coefficient of λ_j is

$$(2j + \tau + 1 + m)^2 \epsilon_j + \sum_i M_i \epsilon_{j+i} + \sum_i N_i \epsilon'_{-j+i} = 0$$

because, since $\Sigma M_i \epsilon_{j+i} = \Sigma M_{-i} \epsilon_{j-i} = \Sigma M_i \epsilon_{j-i}$, this is one of the equations of condition. Similarly all the coefficients on the left-hand side vanish, and the required relation appears in the form

$$0 = \sum_j (A_j \epsilon_j + A'_{-j} \epsilon'_{-j}) \quad \text{.........................(24)}$$

The reason for retaining the terms $(D^2 + 2mD)\, u_1$ in (22) will now be understood. Without them there is no reason why the relation (24) should be satisfied, and in fact it will be contradicted. But let u_1 contain terms of the form

$$(u_1) = \zeta \sum_i (E_i \zeta^{2i+c} + E'_{-i} \zeta^{-2i-c})$$

$$(D^2 + 2mD)(u_1) = \zeta \sum_i \{[c^2 + 2c(2i + 1 + m)]\, E_i \zeta^{2i+c}$$
$$+ [c^2 + 2c(2i - 1 - m)]\, E'_{-i} \zeta^{-2i-c}\}$$

where terms obviously of order less than $\mu + 1$ are omitted. Then clearly, if the value of c here be regarded as unknown, it will be possible to adjust its value so as to satisfy the relation (24).

252. The matter is made clearer by considering the actual facts. In the first order there is one such series, with the coefficients ϵ_i, ϵ_i'. In the second order there is no such series and the question does not arise. The primitive value c_0 suffices. In the third order series of this type reappear, associated with the characteristics e^3, ee'^2, ek^2, $e(\mathbf{a}a'^{-1})^2$. The contemplated change in c is associated with e through the first order terms. Hence the relation (24) in the third order will give in succession the parts of c which contain e^2, e'^2, k^2 and $(\mathbf{a}a'^{-1})^2$. Similarly still higher parts of c may be found in conjunction with the inequalities of a higher order. It is natural that the motion of the perigee (and the value of the characteristic exponent) which was determined for highly simplified conditions, should require adjustment

when the conditions are more complicated and the deviation from the periodic orbit is no longer infinitely small.

For c let $c_1 + \lambda'\delta c$ be written, where $\lambda'\delta c$ is the part to be determined, its characteristic being λ', and let

$$A_j = B_j + D_j \delta c, \quad A'_{-j} = B'_{-j} + D'_{-j} \delta c$$

where B_j, B'_{-j}, D_j, D'_{-j} are calculated numbers. With the new value of c the quantities A_j, A'_{-j} satisfy a certain relation identically as required, and the equations (23) become consistent, but the solution is not definite because any one of the equations can be derived from the rest. An arbitrary condition can be imposed, and the form $\lambda_0' = \lambda_0$ is chosen. The solution is then conducted in the following way.

The equations for $j=0$ are left aside. Three separate solutions are then made of the remaining equations: (1) $\lambda_j = b_j$, $\lambda'_{-j} = b'_{-j}$ when $\lambda_0 = \lambda_0' = 0$ and $A_j = B_j$, $A'_{-j} = B'_{-j}$; (2) $\lambda_j = d_j$, $\lambda'_{-j} = d'_{-j}$ when $\lambda_0 = \lambda_0' = 0$ and $A_j = D_j$, $A'_{-j} = D'_{-j}$; and (3) $\lambda_j = f_j$, $\lambda'_{-j} = f'_{-j}$ when $\lambda_0 = \lambda_0' = 1$ and $A_j = A'_{-j} = 0$. The last, which under the different condition $\lambda_0 - \lambda_0' = 1$ would have led to ϵ_j, ϵ'_{-j}, is independent of A_j, A'_{-j} and applies in all cases. The complete solution is therefore

$$\lambda_j = b_j + d_j \delta c + f_j \lambda_0, \quad \lambda'_{-j} = b'_{-j} + d'_{-j} \delta c + f'_{-j} \lambda_0.$$

When these are inserted in the equations for $j=0$ the result is of the form

$$b_0 + d_0 \delta c + f_0 \lambda_0 = b_0' + d_0' \delta c + f_0' \lambda_0 = 0$$

and δc and λ_0 are thus determined. The value of δc must also satisfy the relation (24), so that a check on the accuracy of the work is provided. The solution of the equations (23) for the case when $\tau = c$ is therefore complete, and the derivation of the higher parts of c has been explained. It may be noted that on the left-hand side of these equations the primitive value c_0 is to be retained for τ at every stage, both because it is associated with terms of the full order $\mu + 1$ and because the theory of the equations depends on the fact that the modulus vanishes. On the other side c will receive its full value so far as it has been determined. When a new part of c comes to be determined in conjunction with inequalities having the characteristic λ, δc is always associated through $(D^2 + 2mD)(u_1)$ with the terms in u_1 of the first order in e. Hence the new part of c itself always has the characteristic $\lambda' = e^{-1}\lambda$, and the numbers d_j, d'_{-j}, like f_j, f'_{-j}, are the same in all cases.

253. With the equation for z matters follow a precisely similar course, and the exceptional case arises when $\tau = g$. The conditions are simpler, because $\lambda_j + \lambda'_{-j} = 0$ always, and therefore the arbitrary relation has the form $\lambda_0 = \lambda_0' = 0$. The terms of the first order with suitable arguments have the characteristic k, and the part of g found in conjunction with inequalities having the characteristic λ contains the characteristic $k^{-1}\lambda$.

The arbitrary condition $\lambda_0 = \lambda_0'$ adopted in all cases has an importance beyond that apparent in the actual calculation. The aggregate of the terms considered up to the final stage of approximation gives for the one argument

$$u = \mathbf{a}\mathrm{e}\zeta(\epsilon_0\zeta^{\mathrm{c}} + \epsilon_0'\zeta^{-\mathrm{c}}) + \mathbf{a}\zeta(\zeta^{\mathrm{c}} + \zeta^{-\mathrm{c}})\Sigma\lambda\lambda_0$$
$$s = \mathbf{a}\mathrm{e}\zeta^{-1}(\epsilon_0\zeta^{-\mathrm{c}} + \epsilon_0'\zeta^{\mathrm{c}}) + \mathbf{a}\zeta^{-1}(\zeta^{\mathrm{c}} + \zeta^{-\mathrm{c}})\Sigma\lambda\lambda_0$$
$$u\zeta^{-1} - s\zeta = \mathbf{a}\mathrm{e}(\epsilon_0 - \epsilon_0')(\zeta^{\mathrm{c}} - \zeta^{-\mathrm{c}}).$$

The last expression remains unaltered throughout the course of the approximations. Hence the constant e is defined as "the coefficient of $\mathbf{a}\sin l$ in the *final* expression of $\rho\sin(v - nt - \epsilon)$ as a sum of periodic terms, where $v - nt - \epsilon$ is the difference of the true and mean longitudes and ρ is the projection of the Moon's radius vector on the plane of reference."

Similarly the terms of the form

$$\iota z = \mathbf{a}\mathrm{k}k_0(\zeta^{\mathrm{g}} - \zeta^{-\mathrm{g}})$$

in the first approximation have no addition made to them subsequently, since $\lambda_0 = \lambda_0' = 0$. Hence the constant k is defined as "the coefficient of $2\mathbf{a}\sin F$ in the (final) expression of z as a sum of periodic terms."

There is no reason to alter the definition of $\mathbf{a}$, which is based on the variational curve. But it is then to be noticed that the constant of distance in the projection on the z plane will no longer be $\mathbf{a}a_0$, where $a_0 = 1$, but will be affected by terms with various characteristics which arise in the course of the approximations as the constant parts of $u\zeta^{-1}$ or $s\zeta$. Either m or $\mathbf{a}$, since they are connected by a certain relation (11), may be regarded as an arbitrary constant of the solution.

The remaining three arbitraries have been denoted by t_0, t_1, t_2. These may be replaced by ϵ, ϖ, θ, the mean longitudes of the Moon and its perigee and node at the epoch $t = 0$. Then

$$\begin{aligned} D &= \phantom{\mathrm{c}}\,(n - n')(t - t_0) = \phantom{\mathrm{c}}\,(n - n')t + \epsilon - \epsilon' \\ l &= \mathrm{c}\,(n - n')(t - t_1) = \mathrm{c}\,(n - n')t + \epsilon - \varpi \\ l' &= \mathrm{m}\,(n - n')(t - t_3) = \phantom{\mathrm{c}}\,n't + \epsilon' - \varpi' \\ F &= \mathrm{g}\,(n - n')(t - t_2) = \mathrm{g}\,(n - n')t + \epsilon - \theta \end{aligned}$$

where ϵ' is the mean longitude of the Sun at the epoch $t = 0$ and ϖ' is the (constant) longitude of the solar perigee. The time t_3 is not an arbitrary: it depends on the Sun alone and is one of the data of the problem.

The formulae for transformation to polar coordinates were given in § 230 for two dimensions only. It is necessary to replace r by ρ, its projection on the plane of the ecliptic, where $\rho^2 = X^2 + Y^2 = us$. Then

$$\begin{aligned} u\zeta^{-1} &= \rho\exp.\,\iota(v - nt - \epsilon) \\ s\zeta &= \rho\exp. - \iota(v - nt - \epsilon) \\ z &= \rho\tan\phi \end{aligned}$$

where ϕ is the latitude. Hence the true longitude and the latitude are

$$v = nt + \epsilon + \tfrac{1}{2}\iota(\log s\zeta - \log u\zeta^{-1})$$

$$\phi = \tan^{-1}\frac{z}{\rho} = \frac{z}{\rho} - \tfrac{1}{3}\left(\frac{z}{\rho}\right)^3 + \tfrac{1}{5}\left(\frac{z}{\rho}\right)^5 - \ldots.$$

The constant of the Moon's horizontal equatorial parallax is based on a, where $n^2a^3 = E + M$. To obtain the parallax at any time this constant must be multiplied by

$$\frac{a}{r} = \frac{a}{\mathbf{a}} \cdot \left(\frac{us + z^2}{\mathbf{a}^2}\right)^{-\frac{1}{2}}.$$

In these expressions for v, ϕ and ar^{-1} the variational parts u_0, s_0 are separated from the other terms u_1, s_1, z, and the expressions are then expanded in terms of the latter. Advantage can thus be taken of the expansions already obtained in the course of the previous work. The conversion to the final form of coordinates therefore entails no great amount of extra labour.

254. This completes in outline the solution of the main part of the problem, in which the Earth, Moon and Sun are treated as centrobaric bodies, and the orbit of the Sun, or the relative orbit of the centre of mass of the Earth-Moon system, is treated as an undisturbed ellipse in a fixed plane. A large number of comparatively small but highly complicated corrections are still necessary in order to represent the gravitational motion of the Moon in actual circumstances. They may be classified thus:

(1) The effect of the ellipsoidal figure of the Earth, and possibly of the Moon.

(2) The direct action of the planets on the relative motion of the Moon.

(3) The indirect action of the planets, which operates by modifying the coordinates of the Sun. These indirect effects are in general larger than the direct effects, and are sometimes sensible in the lunar motion when they are insensible in the relative motion of the Earth and Sun. Among the indirect actions of the planets may be specially mentioned

(4) Lunar inequalities produced by the motion of the ecliptic, and

(5) The secular acceleration of the Moon's mean motion, which arises from the secular change in the solar eccentricity e' under the action of the planets.

It is impossible to discuss these matters profitably in a short space. The reader will find references in Professor Brown's Treatise and detailed results in the memoir* which contains his complete and original theory.

* *Memoirs R. Astr. Soc.*, LIII, pp. 39, 163; LIV, p. 1; LVII, p. 51; LIX, p. 1.

CHAPTER XXII

PRECESSION, NUTATION AND TIME

255. In order to investigate the motion of the Earth about its centre of gravity O we take a set of rectangular axes $OXYZ$ fixed in space and a second set $Oxyz$ coinciding with the principal axes of inertia. These are fixed in the Earth and move with it. The two sets are drawn in such a sense that the positive directions of the corresponding axes can be brought into coincidence by a suitable rotation. Their relative situation is defined by the three Eulerian angles θ, ϕ, ψ, where θ is the angle between OZ and Oz, ϕ is the angle between the planes OXZ and OZz, and ψ is the angle between the planes OZz and Ozx. Then the coordinates are related by the scheme:

	X	Y	Z
x	$\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi$	$\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi$	$-\sin\theta\cos\psi$
y	$-\cos\theta\cos\phi\sin\psi - \sin\phi\cos\psi$	$-\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi$	$\sin\theta\sin\psi$
z	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$

The result of resolving the angular velocities $\dot{\theta}$ which is a rotation in the plane OZz, $\dot{\phi}$ which is a rotation about OZ, and $\dot{\psi}$ which is a rotation about Oz, about Ox, Oy, Oz is to give the equivalent angular velocities about these axes, namely

$$\left.\begin{aligned} \omega_1 &= \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi \\ \omega_2 &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \omega_3 &= \dot{\psi} \qquad\quad + \dot{\phi}\cos\theta \end{aligned}\right\} \qquad \text{.......................(1)}$$

which are Euler's geometrical equations.

Let A, B, C be the moments of inertia about the axes $Oxyz$ and L, M, N the moments of the external forces about these axes. Then the dynamical equations may be written in the well-known form:

$$\left.\begin{aligned} A\dot{\omega}_1 - (B - C)\,\omega_2\omega_3 &= L \\ B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 &= M \\ C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 &= N \end{aligned}\right\} \qquad \text{..........................(2)}$$

256. The external forces which are here considered are due to the action of the Sun and Moon. An approximate expression for the action of either of these bodies is sufficient and easily found. The potential of the Earth (mass m) at a distant point P has been found (§ 18) to be

$$V = G\Sigma \frac{dm}{\rho} = G\left(\frac{m}{r} + \frac{A+B+C-3I}{2r^3}\right)$$

where $OP = r$ and I is the moment of inertia of m about OP. This expression is true as regards terms of the second order in the coordinates of points in m relative to the centre of gravity O. Terms of the third order will clearly vanish in the sum provided that the mass m possesses three rectangular planes of symmetry: and this is sensibly true in the case of the Earth. Terms of the fourth order are small in consequence of the ellipsoidal figure of the Earth and are neglected. Now V is the work done by unit attracting mass at P when the particles of the mass m are brought from infinity to their actual configuration. Hence the work done by a finite mass near a distant point O' is

$$U = G\Sigma\left(\frac{m}{r} + \frac{A+B+C-3I}{2r^3}\right) dm'$$

$$= G\left\{\frac{mm'}{R} + \frac{m(A'+B'+C'-3I')}{2R^3}\right\} + \tfrac{1}{2}G\Sigma \frac{A+B+C-3I}{r^3} dm'$$

by similar reasoning, if O' is the centre of gravity of the attracting mass m', $OO' = R$, A', B', C' are the principal moments of inertia of m' at O' and I' is the moment of inertia of m' about OO'. Now since A, B, C and I are of the second order in the linear dimensions of m, terms of the second order in the linear dimensions of m' can be neglected when associated with them. Let the coordinates of O' relative to O be (x, y, z) and of P relative to O' be (ξ, η, ζ). Then

$$r^2 = (x+\xi)^2 + (y+\eta)^2 + (z+\zeta)^2$$

$$r^2 I = A(x+\xi)^2 + B(y+\eta)^2 + C(z+\zeta)^2.$$

But since O' is the centre of gravity of the mass m'

$$\Sigma\xi dm' = \Sigma\eta dm' = \Sigma\zeta dm' = 0.$$

Hence if the expression to be summed be expanded in terms of ξ, η, ζ, the terms of the first order vanish in the sum and terms of the second order are neglected. To this order of approximation

$$G\Sigma \frac{A+B+C-3I}{r^3} dm' = Gm'\left\{\frac{A+B+C}{R^3} - \frac{3(Ax^2+By^2+Cz^2)}{R^5}\right\}$$

and if I now represents the moment of inertia of m about OO', the complete expression for U becomes

$$U = G\left\{\frac{mm'}{R} + \frac{m(A'+B'+C'-3I')}{2R^3} + \frac{m'(A+B+C-3I)}{2R^3}\right\}.$$

This represents the mutual potential of two masses m, m' with sufficient accuracy. In the usual astronomical units (§ 24) $G = k^2$. The mass of the Sun is unity and for the masses of the Earth and Moon we take E and fE. Then if the mean distances of the Sun and Moon are $a' (= 1)$ and a'' and the mean motions n' and n'',

$$G(1+E) \quad = n'^2 a'^3$$
$$GE(1+f) = n''^2 a''^3.$$

257. The moments of the external forces about the axes $Oxyz$ being L, M, N, the work done by them when the Earth receives a small twist defined by the rotations $d\omega_1$, $d\omega_2$, $d\omega_3$ about the same axes is

$$dU = L\,d\omega_1 + M\,d\omega_2 + N\,d\omega_3.$$

But U depends on the orientation of the Earth only through the occurrence of I; and

$$R^2 I = Ax^2 + By^2 + Cz^2$$

(x, y, z) being the centre of gravity of the attracting body. Hence

$$dU = -3Gm'(Ax\,dx + By\,dy + Cz\,dz)/R^5.$$

But with due regard to sign, when the axes are rotated,

$$dx = y\,d\omega_3 - z\,d\omega_2, \quad dy = z\,d\omega_1 - x\,d\omega_3, \quad dz = x\,d\omega_2 - y\,d\omega_1.$$

Hence, equating the coefficients of $d\omega_1$, $d\omega_2$, $d\omega_3$ in the two expressions for dU,

$$L = 3Gm'(C-B)\,yz/R^5, \quad M = 3Gm'(A-C)\,xz/R^5, \quad N = 3Gm'(B-A)\,xy/R^5.$$

These apply to a body possessing three distinct principal axes. But the Earth may be regarded as an ellipsoid of revolution, for which $B = A$ and $C > A$. Under these circumstances

$$L = 3Gm'(C-A)\,yz/R^5, \quad M = -3Gm'(C-A)\,xz/R^5, \quad N = 0.$$

On the other hand, the term in U which depends on the orientation of the Earth is more generally

$$U' = -\tfrac{3}{2}Gm'I/R^3 = -\tfrac{3}{2}Gm'(Ax^2 + By^2 + Cz^2)/R^5$$
$$= -\tfrac{3}{4}Gm'\{(2C - A - B)z^2 + (A - B)(x^2 - y^2) + (A + B)R^2\}/R^5$$

a useful form for some purposes. The last term on the right, being independent of the orientation, can always be rejected; and when the Earth is considered uniaxal, it is possible to use simply

$$U'' = -\tfrac{3}{2}Gm'(C-A)z^2/R^5 \quad \text{.......................(3)}$$

258. With $B = A$ and $N = 0$, the third equation of (2) gives

$$\dot{\omega}_3 = 0, \quad \omega_3 = n$$

and the other equations of the set become

$$A\dot{\omega}_1 + (C-A)\,n\omega_2 = L$$
$$A\dot{\omega}_2 - (C-A)\,n\omega_1 = M.$$

The actual motion of the Earth is a steady state of rotation disturbed by the external forces and this steady state will be found by putting $L = M = 0$. The equations then give

$$\ddot{\omega}_1 + \mu^2\omega_1 = \ddot{\omega}_2 + \mu^2\omega_2 = 0$$

where

$$\mu = n(C - A)/A.$$

Hence the steady state is given by

$$\omega_1 = h\cos(\mu t + \alpha), \quad \omega_2 = h\sin(\mu t + \alpha).$$

But the instantaneous axis of rotation in the Earth is the line

$$x/\omega_1 = y/\omega_2 = z/\omega_3$$

or

$$x/h\cos(\mu t + \alpha) = y/h\sin(\mu t + \alpha) = z/n$$

which indicates that if h is fairly small the terrestrial pole describes a small circle of radius h/n about the axis of figure in the period $2\pi/\mu$. This is the Eulerian period of $A/(C - A)$ (roughly 300) days. Now the angle between the Zenith of a place and the Pole is the co-latitude of the place, an angle which can be constantly observed. Hence the latitude of any place should exhibit a variation with a period of about 10 months. Until a quarter of a century ago no variation of latitude had certainly been detected. Since that time variations (of the order of $0''\cdot3$) have been systematically observed and studied and have also been traced in the older observations. But analysis has proved conclusively that these variations contain no part which conforms with the Eulerian period. They cannot therefore be explained by the free motion of the Pole on a rigid Earth. Hence observation justifies the belief that h/n is insensibly small.

The variations of latitude observed are always very small and constitute a highly complex phenomenon. The periods of the chief components of the motion of the Pole are about 12 and 14 months.

259. Corresponding to the free movement of the Pole on the Earth's surface we have, by (1),

$$\dot{\theta} = \omega_1\sin\psi + \omega_2\cos\psi = \quad h\sin(\mu t + \alpha + \psi)$$
$$\dot{\phi}\sin\theta = \omega_2\sin\psi - \omega_1\cos\psi = -h\cos(\mu t + \alpha + \psi).$$

For the plane OXY we take the plane of the ecliptic which varies but slightly in consequence of planetary perturbations. The value of θ is about 23°. Hence $\dot{\theta}$ and $\dot{\phi}$ are very small in comparison with n, a fact in accordance with observation even when the disturbing effects of the Sun and Moon are operative. Hence, further, $\dot{\psi}$ differs only slightly from n.

The rotational energy of the Earth is T, where

$$\begin{aligned} 2T &= A(\omega_1^2 + \omega_2^2) + C\omega_3^2 \\ &= A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + C(\dot{\psi} + \dot{\phi}\cos\theta)^2. \end{aligned}$$

Hence the Lagrangian equations of motion are

$$\frac{d}{dt}(A\dot{\theta}) - A\dot{\phi}^2 \sin\theta\cos\theta + C\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) = \frac{\partial U}{\partial\theta}$$

$$\frac{d}{dt}\{A\dot{\phi}\sin^2\theta + C\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta)\} = \frac{\partial U}{\partial\phi}$$

$$\frac{d}{dt}\{C(\dot{\psi} + \dot{\phi}\cos\theta)\} = \frac{\partial U}{\partial\psi}.$$

But since

$$\frac{\partial U}{\partial\psi} = N = 0, \quad \dot{\psi} + \dot{\phi}\cos\theta = n$$

the first two equations become

$$A\ddot{\theta} - A\dot{\phi}^2\sin\theta\cos\theta + Cn\,\dot{\phi}\sin\theta = \frac{\partial U}{\partial\theta}$$

$$\frac{d}{dt}(A\dot{\phi}\sin^2\theta + Cn\cos\theta) = \frac{\partial U}{\partial\phi}.$$

It has been seen that n is very large compared with $\dot{\theta}$ and $\dot{\phi}$, and it follows that those terms are of predominant importance which contain n as a factor. Neglecting the other terms on the left the equations become simply

$$\dot{\phi} = \frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\theta}$$

$$\dot{\theta} = -\frac{1}{Cn\sin\theta}\frac{\partial U}{\partial\phi}.$$

The complete justification for omitting the terms rejected must be sought by substituting in them the results which follow from the latter simple form of equations, when it will be found that they are practically insensible. The form to be used for U is given by (3), so that

$$U = -\tfrac{3}{2}G(C - A)\Sigma m'z^2/R^5$$

a sum of two terms corresponding to the Sun and Moon. For each disturbing body it is necessary to find the product of z^2/R^2 and a^3/R^3 expressed in appropriate terms and with a suitable degree of approximation.

260. The axes XYZ being fixed in space are defined so that OZ is directed towards the pole of the ecliptic for 1850.0 and OX towards the equinox for the same epoch. By the scheme of transformation

$$z = X\sin\theta\cos\phi + Y\sin\theta\sin\phi + Z\cos\theta.$$

The position of a disturbing body, such as the Moon, is more conveniently referred to a similar set of axes for another epoch t. The necessary changes may be considered successively, thus:

(i) Rotate the axes about OZ through the angle Ω so as to bring OX to the position OX_1. Then

$$X = X_1 \cos\Omega - Y_1 \sin\Omega, \quad Y = Y_1 \cos\Omega + X_1 \sin\Omega, \quad Z = Z_1$$

where Ω is the node of the ecliptic for epoch t on the ecliptic for 1850.0.

(ii) Rotate the axes about OX_1 through the angle i so as to bring OY_1 to the position OY_2. Then

$$X_1 = X_2, \quad Y_1 = Y_2 \cos i - Z_2 \sin i, \quad Z_1 = Z_2 \cos i + Y_2 \sin i$$

where i is the inclination of the ecliptic for epoch t to the ecliptic for 1850.0.

(iii) Rotate the axes about OZ_2 through the angle $N - \Omega$ so as to bring OX_2 to the position OX_3. Then

$$X_2 = X_3 \cos(N - \Omega) - Y_3 \sin(N - \Omega),$$
$$Y_2 = Y_3 \cos(N - \Omega) + X_3 \sin(N - \Omega), \quad Z_2 = Z_3$$

where N is the longitude of the Moon's node reckoned through Ω in both ecliptic planes.

(iv) Rotate the axes about OX_3 through the angle c so as to bring OY_3 to the position OY_4. Then

$$X_3 = X_4, \quad Y_3 = Y_4 \cos c - Z_4 \sin c, \quad Z_3 = Z_4 \cos c + Y_4 \sin c$$

where c is the inclination of the Moon's orbit to the ecliptic for epoch t.

But, if (X_4, Y_4, Z_4) are the Moon's coordinates,

$$X_4 = r\cos(v - N), \quad Y_4 = r\sin(v - N), \quad Z_4 = 0$$

where r is the radius vector and v is the longitude of the Moon at epoch t reckoned in its orbit; this longitude is the sum of three arcs in the two ecliptic planes and the plane of the lunar orbit. Now $i < 1°$ and, for the Moon, c is of the order 5°. Terms of the order i^2, c^3 and ic are therefore neglected. Then the result of eliminating (X_3, Y_3, Z_3), (X_4, Y_4, Z_4) gives

$$X_2 = r\cos(v - \Omega) + \tfrac{1}{2}c^2 r \sin(v - N)\sin(N - \Omega)$$
$$Y_2 = r\sin(v - \Omega) - \tfrac{1}{2}c^2 r \sin(v - N)\cos(N - \Omega)$$
$$Z_2 = cr\sin(v - N)$$

and the result of eliminating (X, Y, Z), (X_1, Y_1, Z_1) gives

$$z = X_2 \sin\theta\cos(\phi - \Omega) + Y_2 \sin\theta\sin(\phi - \Omega) + Z_2\cos\theta$$
$$+ i\{Y_2\cos\theta - Z_2\sin\theta\sin(\phi - \Omega)\}.$$

Hence

$$z/r = \sin\theta\cos(v - \phi) + c\cos\theta\sin(v - N) - \tfrac{1}{2}c^2\sin\theta\sin(v - N)\sin(\phi - N)$$
$$+ i\cos\theta\sin(v - \Omega).$$

In squaring this expression terms not involving θ or ϕ can be rejected, because they disappear on differentiation. Also terms involving v with

coefficients above zero order are found to be negligible in effect. Under these conditions the result becomes

$$z^2/r^2 = \tfrac{1}{2}\sin^2\theta + \tfrac{1}{2}\sin^2\theta\cos 2(v-\phi)$$
$$+ c\sin\theta\cos\theta\sin(\phi - N) + i\sin\theta\cos\theta\sin(\phi-\Omega)$$
$$+ \tfrac{1}{4}c^2\sin^2\theta\cos 2(\phi - N) - \tfrac{3}{4}c^2\sin^2\theta \quad\ldots\ldots\ldots\ldots(4)$$

261. Certain expansions in terms of the mean anomaly in undisturbed elliptic motion are now required. When e^3 is neglected in the formulae of § 40, (22), (26) and (27) of Chapter IV become

$$r/a = 1 + \tfrac{1}{2}e^2 - e\cos M - \tfrac{1}{2}e^2\cos 2M$$
$$a^2x/r^3 = (1 - \tfrac{9}{8}e^2)\cos M + 2e\cos 2M + \tfrac{27}{8}e^2\cos 3M$$
$$a^2y/r^3 = (1 - \tfrac{5}{8}e^2)\sin M + 2e\sin 2M + \tfrac{27}{8}e^2\sin 3M.$$

The latter give, w being the true anomaly,

$$a^4\sin 2w/r^4 = (1-e^2)\sin 2M + 4e\sin 3M + \tfrac{43}{4}e^2\sin 4M$$
$$a^4\cos 2w/r^4 = \tfrac{1}{4}e^2 + (1-e^2)\cos 2M + 4e\cos 3M + \tfrac{43}{4}e^2\cos 4M$$
$$a^4/r^4 = 1 + 3e^2 + 4e\cos M + 7e^2\cos 2M$$

whence, after multiplication by r/a,

$$a^3\sin 2w/r^3 = [-\tfrac{1}{2}e\sin M] + (1-\tfrac{5}{2}e^2)\sin 2M + [\tfrac{7}{2}e\sin 3M + \tfrac{17}{2}e^2\sin 4M]$$
$$a^3\cos 2w/r^3 = [-\tfrac{1}{2}e\cos M] + (1-\tfrac{5}{2}e^2)\cos 2M + [\tfrac{7}{2}e\cos 3M + \tfrac{17}{2}e^2\cos 4M]$$
$$a^3/r^3 = 1 + \tfrac{3}{2}e^2 + 3e\cos M + [\tfrac{9}{2}e^2\cos 2M].$$

The eccentricity being small, of the same order as c, the terms [ ] which involve M and are not of zero order, are immediately rejected. Now

$$M = n''t + \mu - \varpi$$
$$v = w + \varpi$$

where $n''t + \mu$ is the mean longitude of the Moon in its orbit and ϖ is the longitude of the lunar perigee, both being measured partly in the two ecliptic planes for 1850·0 and the epoch t and partly in the plane of the lunar orbit. From the expression (4) can now be derived

$$a^3z^2/r^5 = (\tfrac{1}{2} - \tfrac{3}{4}c^2 + \tfrac{3}{4}e^2)\sin^2\theta + c\sin\theta\cos\theta\sin(\phi - N)$$
$$+ i\sin\theta\cos\theta\sin(\phi - \Omega) + \tfrac{1}{4}c^2\sin^2\theta\cos 2(\phi - N)$$
$$+ \tfrac{1}{2}\sin^2\theta\cos 2(n''t + \mu - \phi) + \tfrac{3}{2}e\sin^2\theta\cos(n''t + \mu - \varpi)$$

the final term being retained though periodic and not of zero order.

For the Sun $c = 0$ and hence similarly

$$a'^3z'^2/r'^5 = (\tfrac{1}{2} + \tfrac{3}{4}e'^2)\sin^2\theta + i\sin\theta\cos\theta\sin(\phi - \Omega)$$
$$+ \tfrac{1}{2}\sin^2\theta\cos 2(n't + \mu' - \phi) + \tfrac{3}{2}e'\sin^2\theta\cos(n't + \mu' - \varpi').$$

262. These expressions give the means of forming U, for

$$U = -\tfrac{3}{2}G(C-A)\Sigma m'z^2/R^5.$$

For the Moon (§ 256)

$$\frac{Gm'}{a^3} = \frac{GEf}{a''^3} = \frac{fn''^2}{1+f}$$

and for the Sun

$$\frac{Gm'}{a^3} = \frac{G}{a'^3} = \frac{n'^2}{1+E}.$$

Let

$$K_2 = \tfrac{3}{2}\cdot\frac{C-A}{Cn}\cdot\frac{fn''^2}{1+f},\quad K_1 = \tfrac{3}{2}\cdot\frac{C-A}{Cn}\cdot\frac{n'^2}{1+E} \quad\text{............(5)}$$

Then

$$\frac{U}{Cn} = -K_2\cdot\frac{a^3z^2}{r^5} - K_1\cdot\frac{a'^3z'^2}{r'^5}$$

$$\begin{aligned}
&= -\{K_2(\tfrac{1}{2}-\tfrac{3}{4}c^2+\tfrac{3}{4}e^2)+K_1(\tfrac{1}{2}+\tfrac{3}{4}e'^2)\}\sin^2\theta - \tfrac{1}{2}(K_1+K_2)\,i\sin 2\theta\sin(\phi-\Omega)\\
&\quad - K_1\{\tfrac{1}{2}\cos 2(n't+\mu'-\phi)+\tfrac{3}{2}e'\cos(n't+\mu'-\varpi')\}\sin^2\theta\\
&\quad - K_2\{\tfrac{1}{2}\cos 2(n''t+\mu-\phi)+\tfrac{3}{2}e\cos(n''t+\mu-\varpi)\}\sin^2\theta\\
&\quad - K_2\{c\sin\theta\cos\theta\sin(\phi-N)+\tfrac{1}{4}c^2\sin^2\theta\cos 2(\phi-N)\} \quad\text{..................(6)}
\end{aligned}$$

The dynamical equations (§ 259)

$$\dot\phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{U}{Cn}\right)$$

$$\dot\theta = -\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{U}{Cn}\right)$$

which result must be solved by continual approximation. This process, when guided by the facts of observation and limited to practical requirements for a period of a century or two, is very simple. For it is known that θ is very nearly constant, while ϕ changes progressively but very slowly. Hence it is possible to discuss the secular effects, or precession, and the periodic effects, or nutation, separately.

263. The last three lines in the expression for U/Cn, containing six terms, give rise to periodic terms in $\dot\theta$, $\dot\phi$, which can be neglected in the first instance. The secular changes come from the terms in the first line. With sufficient accuracy we may write

$$i\sin\Omega = gt,\quad i\cos\Omega = g't,\quad e' = e_0 + e_1t$$

the quantities e_0, e_1, g and g' being given by the theory of the Sun's motion. The corresponding changes for the Moon are negligible in effect or rather are treated differently. Hence the equations for the secular movements of the Earth's axis are

$$\begin{aligned}
\dot\phi &= -\{K_2(1-\tfrac{3}{2}c^2+\tfrac{3}{2}e^2)+K_1(1+\tfrac{3}{2}e_0^2)\}\cos\theta\\
&\quad -(K_1+K_2)\frac{\cos 2\theta}{\sin\theta}(g'\sin\phi - g\cos\phi)\,t - 3K_1e_0e_1\,.\,t\cos\theta\\
\dot\theta &= \quad (K_1+K_2)\cos\theta\,(g'\cos\phi+g\sin\phi)\,t.
\end{aligned}$$

When $t=0$ (1850·0), θ is the mean obliquity of the ecliptic for that date and may be denoted by ϵ_0. Also ϕ, being the angle between the planes OXZ and OZz (§ 255), is 90° by the definition of the axis OX. The periodic effects at the time $t=0$ are excluded from consideration here, but their influence is small. Hence initially

$$\left.\begin{aligned}\phi &= 90^\circ - \{K_2(1-\tfrac{3}{2}c^2+\tfrac{3}{2}e^2)+K_1(1+\tfrac{3}{2}e_0^2)\}\cos\epsilon_0\,.\,t\\ &\quad -\left\{\tfrac{1}{2}(K_1+K_2)\frac{\cos 2\epsilon_0}{\sin\epsilon_0}g'+\tfrac{3}{2}K_1e_0e_1\cos\epsilon_0\right\}t^2\\ \theta &= \epsilon_0+\tfrac{1}{2}(K_1+K_2)\cos\epsilon_0\,.\,gt^2\end{aligned}\right\}\quad\ldots\ldots(7)$$

The length of time during which these expressions will be valid depends on the numerical values of the quantities involved. For a short interval from 1850·0 (a century or two) the preceding equations hold good, and may be written

$$\left.\begin{aligned}\phi_m &= 90^\circ-\alpha t-\beta t^2\\ \theta_m &= \epsilon_0+\gamma t^2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

the suffix m denoting mean values from which periodic changes are excluded. Thus ϕ_m, θ_m define the position of the *mean* equator at the time t relative to the fixed ecliptic (1850·0), the coefficients α, β and γ being now determined by (7). The motion of the mean equator on the fixed ecliptic, measured by $90^\circ-\phi_m$, is called the *luni-solar precession* in longitude. The angle $\theta_m-\epsilon_0$ may be called the luni-solar precession in obliquity.

264. It has been convenient to use a fixed set of axes XYZ, where Z represents the pole of the ecliptic for 1850·0 and X the mean equinox for the same date. It is now necessary to introduce a new set of axes $X'Y'Z'$, where Z' represents the pole of the ecliptic for the epoch t and X' the corresponding mean equinox, i.e. the intersection of the mean equator and ecliptic at the epoch t. Let z represent the N. pole of this mean equator, its position being defined by ϕ_m, θ_m. The longitude of Z' in the XYZ system is $\Omega-90^\circ$ and $ZZ'=i$, where

$$i\sin\Omega = gt\ +ht^2$$

$$i\cos\Omega = g't+h't^2$$

the terms of the second order being omitted above because they clearly give rise to terms of the third order only in the luni-solar precessions.

Let us consider the spherical triangle $ZZ'z$, of which two sides are $ZZ'=i$ and $Zz=\theta_m$. Since $XZZ'=\Omega-90^\circ$ and $XZz=\phi_m$, the angle $Z'Zz=\phi_m-\Omega+90^\circ$. The side zZ', which is the *mean obliquity of the ecliptic* at t, will be denoted by θ_m', and the angle ZzZ', which is called the *planetary precession*, will be denoted by a. Hence

$$\cot i\sin\theta_m = \cos\theta_m\sin(\Omega-\phi_m)+\cot a\cos(\Omega-\phi_m)$$

and to the second order

$$a = \frac{i \cos(\Omega - \phi_m)}{\cos i \sin \theta_m - i \sin(\Omega - \phi_m) \cos \theta_m}$$

$$= \frac{(g't + h't^2) \cos \phi_m + (gt + ht^2) \sin \phi_m}{\sin \theta_m - \{(gt + ht^2) \cos \phi_m - (g't + h't^2) \sin \phi_m\} \cos \theta_m}$$

$$= \frac{\alpha g't^2 + gt + ht^2}{\sin \epsilon_0 + g't \cos \epsilon_0}$$

since it is enough to take $\theta_m = \epsilon_0$ and $\phi_m = 90° - \alpha t$. Hence to the required order

$$a = \frac{gt}{\sin \epsilon_0} + \frac{t^2}{\sin \epsilon_0}(h + \alpha g' - gg' \cot \epsilon_0) \quad \text{.................(9)}$$

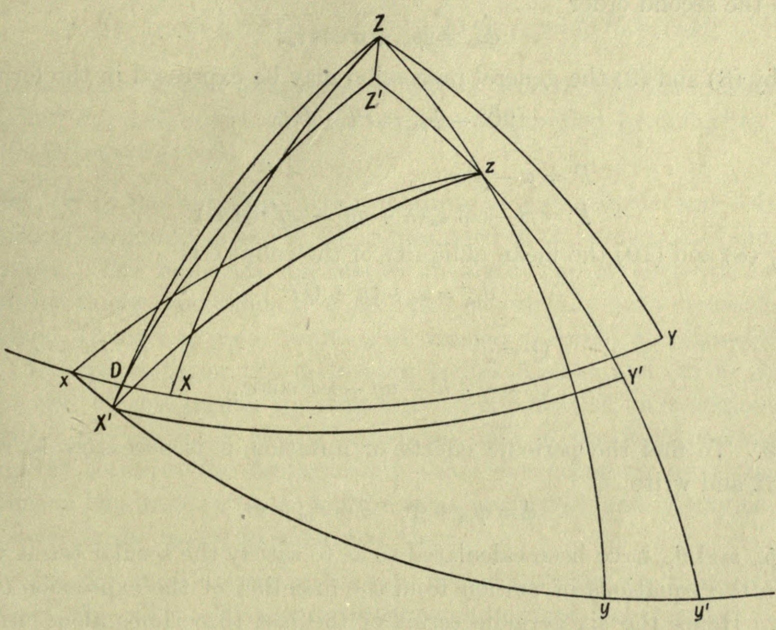

Fig. 8.

Again, in the same triangle,

$$\cos \theta_m' = \cos i \cos \theta_m + \sin i \sin \theta_m \sin(\Omega - \phi_m)$$

whence, to the second order,

$$(\theta_m - \theta_m') \sin \tfrac{1}{2}(\theta_m + \theta_m') = -\tfrac{1}{2} i^2 \cos \theta_m + \sin \theta_m (\alpha g t^2 - g't - h't^2).$$

To the first order, therefore,

$$\theta_m - \theta_m' = -g't, \quad \sin \tfrac{1}{2}(\theta_m + \theta_m') = \sin \epsilon_0 + \tfrac{1}{2} g't \cos \epsilon_0.$$

Hence to the second order

$$\theta_m' - \theta_m = \frac{\frac{1}{2}(g^2+g'^2)\,t^2\cos\epsilon_0 + (g't + h't^2 - \alpha g t^2)\sin\epsilon_0}{\sin\epsilon_0 + \frac{1}{2}g't\cos\epsilon_0}$$

$$= g't + h't^2 - \alpha g t^2 + \tfrac{1}{2}g^2t^2\cot\epsilon_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

The relations between the various sets of axes are shown in fig. 8. The equator $X'y$ (epoch t) cuts the fixed ecliptic XY in x, where $Xx = zZY = 90° - \phi_m$, the luni-solar precession, and $xX' = xzX' = ZzZ' = a$, the planetary precession. Let ZX' cut XY in D, so that XD is the negative mean longitude (1850·0) of X', the mean equinox at t. This arc is called the *general precession* and will be denoted by $90° - \phi_m'$, so that $xD = \phi_m' - \phi_m$. The angle $DxX' = Zz = \theta_m$ and xDX' is a right angle. Hence

$$\tan(\phi_m' - \phi_m) = \tan a \cos\theta_m$$

and to the second order

$$\phi_m' = \phi_m + a\cos\epsilon_0.$$

Thus by (8) and (9) the general precession may be expressed in the form

$$90° - \phi_m' = Pt + P't^2$$

where

$$P = \alpha - g\cot\epsilon_0$$

$$P' = \beta - \cot\epsilon_0\,(h + \alpha g' - gg'\cot\epsilon_0)$$

and by (8) and (10) the mean obliquity of the ecliptic is

$$\theta_m' = \epsilon_0 + Qt + Q't^2$$

where

$$Q = g'$$

$$Q' = \gamma + h' - \alpha g + \tfrac{1}{2}g^2\cot\epsilon_0.$$

265. To find the periodic effects, or nutation, it is necessary to return to § 262 and write

$$\phi = \phi_m + \Phi, \quad \theta = \theta_m + \Theta.$$

Now ϕ_m and θ_m have been calculated so as to satisfy the secular terms which arise in the equations of motion from the first line of the expression (6) for U/Cn. Hence the six periodic terms of the last three lines alone are now relevant, and the dynamical equations become

$$\begin{aligned}\dot\Phi = &- K_1\{\cos 2(n't + \mu' - \phi) + 3e'\cos(n't + \mu' - \varpi')\}\cos\theta \\ &- K_2\{\cos 2(n''t + \mu - \phi) + 3e\cos(n''t + \mu - \varpi)\}\cos\theta \\ &- K_2\{c\sin(\phi - N)\cos 2\theta/\sin\theta + \tfrac{1}{2}c^2\cos 2(\phi - N)\cos\theta\}\end{aligned}$$

$$\begin{aligned}\dot\Theta = &\{K_1\sin 2(n't + \mu' - \phi) + K_2\sin 2(n''t + \mu - \phi)\}\sin\theta \\ &+ K_2\{c\cos\theta\cos(\phi - N) - \tfrac{1}{2}c^2\sin\theta\sin 2(\phi - N)\}.\end{aligned}$$

The Moon's node makes a circuit of the ecliptic in $18\frac{2}{3}$ years in the retrograde direction, so that it is possible to write

$$N = N_0 - N_1 t.$$

To the first order in t, which is alone necessary, $\theta = \epsilon_0$ and $\phi = 90° - \alpha t$; the coefficient α can clearly be incorporated with n', n'' and N_1 before integration in those terms in which ϕ occurs, though the change in n', n'' is unimportant. Then on integration

$$\Phi = K_1 \cos\epsilon_0 \left\{\frac{1}{2n'} \sin 2(n't + \mu') - \frac{3e_0}{n'} \sin(n't + \mu' - \varpi')\right\}$$

$$+ K_2 \cos\epsilon_0 \left\{\frac{1}{2n''} \sin 2(n''t + \mu) - \frac{3e}{n''} \sin(n''t + \mu - \varpi)\right\}$$

$$+ K_2 \left\{\frac{c}{N_1} \sin(N_0 - N_1 t) \cos 2\epsilon_0 / \sin\epsilon_0 - \frac{c^2}{4N_1} \sin 2(N_0 - N_1 t) \cos\epsilon_0\right\}$$

$$\Theta = \sin\epsilon_0 \left\{\frac{K_1}{2n'} \cos 2(n't + \mu') + \frac{K_2}{2n''} \cos 2(n''t + \mu)\right\}$$

$$+ K_2 \left\{\frac{c}{N_1} \cos\epsilon_0 \cos(N_0 - N_1 t) - \frac{c^2}{4N_1} \sin\epsilon_0 \cos 2(N_0 - N_1 t)\right\}.$$

It is unnecessary to add integration constants because these are incorporated in ϕ_m and θ_m, and, except as so far explained, annulled by definition at the initial epoch $t = 0$ (1850).

266. Θ is the nutation of the obliquity of the ecliptic, and $-\Phi$ is the nutation of longitude, ϕ and Φ being measured in the direction of increasing longitudes. The numerical quantities involved are of such an order of magnitude that a fair standard of accuracy has already been obtained in the formulae. If more precise results were needed, it would be necessary (1) to carry the expansions for the disturbing bodies further, and (2) to continue the process of integration by successive approximation to a higher stage. The latter process would clearly introduce terms of the form $at \sin(nt + \alpha)$. Among the terms of the former origin those depending on three times the Sun's mean longitude $(n't + \mu')$ are the most important, and it may be left as an exercise to the reader to determine them.

By far the most important terms in the nutation are those with the argument $(N_0 - N_1 t)$. The other terms being omitted, let

$$\mathcal{N} = K_2 c \cos\epsilon_0 / N_1 \quad \text{.............................(11)}$$

$$x = [\Phi] \sin\epsilon_0 = \mathcal{N} \sin(N_0 - N_1 t) \cos 2\epsilon_0 / \cos\epsilon_0$$

$$y = -[\Theta] = -\mathcal{N} \cos(N_0 - N_1 t).$$

Since $\mathcal{N}$ is an angle of a few seconds only, x and y may be considered as the rectangular plane coordinates of the Earth's pole relative to the mean pole, x being measured in the direction of increasing longitudes and y upwards towards the pole of the ecliptic. The relative path of the true pole is therefore the small ellipse

$$x^2 \cos^2\epsilon_0 + y^2 \cos^2 2\epsilon_0 = \mathcal{N}^2 \cos^2 2\epsilon_0$$

described in a period of about 18 years. Since $\cos \epsilon_0 > \cos 2\epsilon_0$ the major axis is directed towards the pole of the ecliptic and, since $\dot{x}$ has the same sign as y, the sense of description is such that the relative longitude of the true pole is increasing when it lies between the mean pole and the pole of the ecliptic, that is, it is clockwise when viewed from a point outside the celestial sphere. The centre of this elliptic motion is carried by precession almost uniformly in the direction of decreasing longitudes round the pole of the ecliptic.

267. Since the manner of the investigation has been controlled by the actual magnitude of the various quantities involved, it is necessary to introduce numerical values if the results are to be properly understood. Three quantities are based on observation, and not derived from theory, namely, the obliquity ϵ_0 at the fundamental epoch 1850·0, the precession constant P and the nutation constant $\mathcal{N}$. The values now accepted are

$$\epsilon_0 = 23^\circ\, 27'\, 31''\cdot 7, \quad P = 50''\cdot 2453, \quad \mathcal{N} = 9''\cdot 210.$$

The eccentricity of the Earth's orbit is given by

$$e' = e_0 + e_1 t = 0\cdot 016\,7719 - 0\cdot 000\,000418\, t$$

and the position of the ecliptic by

$$i \sin \Omega = gt + ht^2 = +\,0''\cdot 05341\, t + 0''\cdot 000\,01935\, t^2$$
$$i \cos \Omega = g't + h't^2 = -\,0''\cdot 46838\, t + 0''\cdot 000\,00563\, t^2$$

the unit of time being a Julian year of 365·25 mean solar days. The Sun's period relative to the equinox is the tropical year, and the corresponding mean motion is therefore

$$n' = 2\pi \times 365\cdot 25/365\cdot 2422 = 6\cdot 28332.$$

The eccentricity and inclination of the Moon's orbit are

$$e = 0\cdot 05490, \quad c = 5^\circ\, 8'\, 43'' = 0\cdot 089802.$$

The tropical period of the Moon is 27·32158 days, and hence the mean motion in a Julian year is

$$n'' = 83\cdot 997 \text{ radians.}$$

The retrograde motion of the Moon's node has a sidereal period of 6793·5 days. The corresponding mean motion, corrected for precession, is

$$N_1 = 0\cdot 33757 \text{ radians.}$$

It is now possible to derive the values of K_1 and K_2. In the first place, by (11),

$$K_2 = \mathcal{N} N_1 / c \cos \epsilon_0 = 37''\cdot 74.$$

Also

$$\alpha = P + g \cot \epsilon_0 = 50''\cdot 2453 + 0''\cdot 1231 = 50''\cdot 3684.$$

But, by (7) and (8),

$$\alpha \sec \epsilon_0 = K_2 (1 - \tfrac{3}{2} c^2 + \tfrac{3}{2} e^2) + K_1 (1 + \tfrac{3}{2} e_0^2)$$

whence

$$54''\cdot91 = 0\cdot992425\,K_2 + 1\cdot000422\,K_1$$

and thus

$$K_1 = 17''\cdot45.$$

Since any error in $\mathcal{N}$ affects K_2 directly and hence K_1 equally, greater accuracy would be superfluous. The expressions for the luni-solar precession (§ 263) now become

$$90° - \phi_m = \alpha t + \beta t^2 = 50''\cdot3684\,t - 0''\cdot000\,1077\,t^2$$

$$\theta_m = \epsilon_0 + \gamma t^2 = 23°\,27'\,31''\cdot7 + 0''\cdot000\,0066\,t^2$$

while the general precession (§ 264) becomes

$$90° - \phi_m' = Pt + P't^2 = 50''\cdot2453\,t + 0''\cdot000\,1107\,t^2$$

and the mean obliquity of the ecliptic

$$\theta_m' = \epsilon_0 + Qt + Q't^2$$

$$= 23°\,27'\,31''\cdot7 - 0''\cdot46838\,t - 0''\cdot000\,0008\,t^2.$$

268. In giving the numerical values of the terms in the nutation (§ 265) the notation is changed to that employed in the *Nautical Almanac*. The results which follow from substituting the above constants are:

$$\Phi = +\,17''\cdot23 \sin ☊ - 0''\cdot21 \sin 2☊ + 1''\cdot27 \sin 2L$$

$$- 0''\cdot13 \sin (L - \pi) + 0''\cdot21 \sin 2☾ - 0''\cdot07 \sin g_1$$

$$\Theta = +\,9''\cdot21 \cos ☊ - 0''\cdot09 \cos 2☊ + 0''\cdot55 \cos 2L + 0''\cdot09 \cos 2☾$$

where L is the Sun's mean longitude $(n't + \mu')$, π is the longitude of the Sun's perigee (ϖ'), ☾ is the Moon's mean longitude $(n''t + \mu)$, g_1 is the Moon's mean anomaly $(n''t + \mu - \varpi)$, and ☊ is the longitude of the Moon's ascending node $(N_0 - N_1 t)$. In the *Nautical Almanac* the nutation of the obliquity of the ecliptic (Θ) is called $\Delta\omega$, and the nutation of longitude $(-\Phi)$ is called ΔL. Comparison shows that no term with coefficient exceeding $0''\cdot05$ has been omitted here.

Two important astronomical constants are involved implicitly in the constants of nutation and precession, namely the mass of the Moon and the ratio $(C-A)/C$, which has been called the mechanical ellipticity of the Earth. For the equations (5) may be written

$$\frac{f}{1+f} = \frac{K_2}{K_1} \cdot \frac{n'^2}{n''^2}, \quad \frac{C-A}{C} = \frac{2}{3} \cdot \frac{nK_1}{n'^2}$$

the mass of the Earth, $E = 1/333432$, being negligible. Here K_1 and K_2, expressed above in seconds of arc, are angular motions in a Julian year, and n, n' and n'' are sidereal mean motions in the same unit of time. With sufficient accuracy the above values of n' and n'' may be used, and for n the value $2\pi \times 366\tfrac{1}{4}$. Hence

$$f/(1+f) = 0\cdot012102, \quad f = 1/81\cdot6$$

for f, the ratio of the mass of the Moon to the mass of the Earth, and

$$\frac{C-A}{C}=\frac{1}{304 \cdot 2}$$

for the mechanical ellipticity of the Earth. The mass of the Moon is also obtained as a by-product from the observations of a minor planet in a refined determination of the solar parallax. The value of f found by Hinks in this way was $1/81 \cdot 53$.

269. The practical application of the results obtained for precession and nutation belongs to the domain of Spherical Astronomy and will not be pursued in detail here. Nutation is so small that its effects can be, and are, treated independently of those due to precession. Of the latter something more may be said in order to define the two quantities employed in calculating the effects of precession in right ascension and declination.

Let α, δ be the R.A. and declination of a star at the epoch t. These refer to the system of axes $X'y'z$ (fig. 8), which differs by a simple rotation through the angle a about z from the system xyz. Hence the coordinates of the star in the latter system are

$$x=\cos\delta\cos(\alpha+a), \quad y=\cos\delta\sin(\alpha+a), \quad z=\sin\delta$$

whence, by differentiation with respect to t, it easily follows that

$$\dot{\alpha}+\dot{a}=(x\dot{y}-y\dot{x})/\cos^2\delta$$
$$\dot{\delta}=\dot{z}/\cos\delta.$$

Now the relations between the systems xyz and XYZ are expressed by the scheme:

$$\begin{array}{cccc} & X & Y & Z \\ x & \sin\phi & -\cos\phi & 0 \\ y & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ z & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta. \end{array}$$

Here XYZ are constant, and differentiation of the linear formulae for xyz, when XYZ are finally expressed in terms of x, y, z, gives

$$\dot{x}=(y\cos\theta+z\sin\theta)\,\dot{\phi}$$
$$\dot{y}=-x\cos\theta\,.\,\dot{\phi}-z\dot{\theta}$$
$$\dot{z}=-x\sin\theta\,.\,\dot{\phi}+y\dot{\theta}.$$

Hence, when x, y, z are expressed in terms of α, δ,

$$\dot{\alpha}+\dot{a}=-\cos\theta\,.\,\dot{\phi}-\tan\delta\sin(\alpha+a)\sin\theta\,.\,\dot{\phi}-\tan\delta\cos(\alpha+a)\,.\,\dot{\theta}$$
$$\dot{\delta}=-\cos(\alpha+a)\sin\theta\,.\,\dot{\phi}+\sin(\alpha+a)\,\dot{\theta}.$$

These differential expressions are required to the first order in t, and $a\dot{\theta}$ being of the second order may be rejected at once. Hence (the symbol n being used here in a new sense)

$$\dot{\alpha} = m + n \sin\alpha \tan\delta - p \cos\alpha \tan\delta$$

$$\dot{\delta} = n \cos\alpha + p \sin\alpha$$

where

$$m = -\dot{a} - \cos\theta \,.\, \dot{\phi}, \quad n = -\sin\theta \,.\, \dot{\phi}, \quad p = a \sin\theta \,.\, \dot{\phi} + \dot{\theta}$$

and θ may be replaced by ϵ_0. With the numerical values given in § 267, (9) gives

$$a = +0''\cdot1342\, t - 0''\cdot000\,2380\, t^2$$

$$\dot{a} = +0''\cdot1342 \quad - 0''\cdot000\,4760\, t$$

and from the luni-solar precessions

$$\dot{\phi} = -50''\cdot3684 + 0''\cdot000\,2154\, t$$

$$\dot{\theta} = \qquad\qquad + 0''\cdot000\,0132\, t.$$

Hence

$$m = +46''\cdot0711 + 0''\cdot000\,2784\, t$$

$$n = +20''\cdot0511 - 0''\cdot000\,0857\, t$$

while $p = +0''\cdot000\,0002$ and is altogether negligible. Thus m and n are the important quantities known as the *annual precessions* in R.A. and declination. The total precession in R.A. from 1850 for a point on the equator is

$$\int_0 m\,dt = m_1 t + m_2 t^2 = 46''\cdot0711\, t + 0''\cdot000\,1392\, t^2.$$

The expressions found for $\dot{\alpha}$, $\dot{\delta}$ are the coefficients of the first power of the time and these terms suffice for short intervals only. The further development of formulae for the transformation of coordinates from one epoch to another according to the methods of astronomical practice must be sought in such works as Newcomb's *Compendium of Spherical Astronomy.*

270. It is now possible to consider in some detail the astronomical measure of time. The third equation of (1) is

$$\omega_3 = \dot{\psi} + \dot{\phi} \cos\theta.$$

Here ω_3 is the angular velocity of the Earth about its axis of figure and is a constant previously denoted by n. As this symbol has been used with another meaning in § 269 it will now be replaced by ω. The angle ψ is the angle between a meridian plane (Ozx) fixed in the Earth and rotating with it and the plane (OZz) passing through the pole of the fixed ecliptic. For the fixed meridian we adopt the meridian of Greenwich. The rotation $\dot{\psi}$ refers therefore to the Greenwich meridian relative to zx in fig. 8, and $\dot{\tau} = \dot{\psi} - \dot{a}$ will measure the same rotation relative to zx'. But the angle between the Greenwich meridian and zx', x' being the equinoctial point at the time t, is the hour-angle of the First Point of Aries, i.e. the *sidereal time* at Greenwich. Thus, τ being Greenwich sidereal time,

$$\dot{\tau} = \dot{\psi} - \dot{a} = \omega - \dot{a} - \dot{\phi} \cos\theta.$$

It is the true equinox which is now involved, affected both by precession and nutation, so that

$$\phi = \phi_m + \Phi, \quad \theta = \theta_m + \Theta.$$

Hence

$$\begin{aligned}\dot{\tau} &= \omega - \dot{a} - \dot{\phi}_m \cos\theta_m - \dot{\Phi}\cos\theta + \dot{\phi}_m \Theta \sin\theta_m \\ &= \omega + m - \dot{\Phi}\cos\theta - n\Theta \\ &= \omega + m - \dot{\Phi}\cos\epsilon_0\end{aligned}$$

with sufficient accuracy, for $n\Theta$ can be neglected since Θ is small and n is about 10^{-4}, and $\dot{\Phi}$ being small $\cos\theta$ may be replaced by $\cos\epsilon_0$. Hence integration gives for Greenwich sidereal time

$$\tau = \tau_0 + \omega t + m_1 t + m_2 t^2 - \Phi\cos\epsilon_0 \quad\text{..................(12)}$$

where t is measured in Julian years of 365·25 mean days and reckoned from 1850 Jan. 0, Gr. mean noon. The quantity t is an equi-crescent variable in the sense required by the dynamical laws which have been used; its origin and unit are for the moment of importance only so far as they condition the numerical values of the coefficients. On the other hand the sidereal time τ is not uniform, being affected by secular and periodic terms. Hence τ is merely an intermediate standard of time. But this in no way affects its practical utility. By far the largest term in $\Phi\cos\epsilon_0$ is

$$15''\!\cdot\!803 \sin\Omega = 1^s\!\cdot\!054 \sin\Omega$$

of which the period is nearly 19 years, and m_2 is very small. The irregularities in τ are therefore very small and gradual, and far less than the natural irregularities in the rate of the most perfect sidereal clock. Since this instrument shows the hour-angle of the First Point of Aries, it also shows the right ascension of stars on the meridian, and this principle serves both to determine the error of the clock and to measure the apparent positions of the stars.

271. In the next place a *mean Sun* is defined which moves in the plane of the equator with the uniform sidereal mean motion μ. Its R.A. at time t, reckoned from the true equinox, is therefore

$$A = A_0 + \mu t + m_1 t + m_2 t^2 - \Phi\cos\epsilon_0$$

and its hour-angle

$$T = \tau - A = \tau_0 - A_0 + (\omega - \mu)\,t$$

is the measure of Greenwich *mean time.* The constants occurring in A are adjusted as far as possible to secure identity with the mean longitude of the actual Sun affected by aberration. This may be written in the form

$$\begin{aligned}L &= (\lambda_0 + \lambda_1 t + \lambda_2 t^2) - k + (Pt + P't^2) \\ &= L_0 + L_1 t + L_2 t^2\end{aligned}$$

where λ_0 is the true mean longitude of the Sun when $t=0$, λ_1 is the sidereal mean motion, and $2\lambda_2$ is the secular acceleration which arises indirectly from the perturbations of the other elements of the Earth's orbit; $k=20''\cdot47$ is the constant of aberration; and $(Pt+P't^2)$ is the general precession in longitude. The adjustment of the constants in A and L gives

$$A_0=L_0, \quad \mu+m_1=L_1$$

and leaves outstanding between L and A the secular discrepancy $(L_2-m_2)\,t^2$ which would lead ultimately to a departure of the actual Sun, apart from periodic effects, from the meridian at mean noon. This quantity is small and far from certain in amount, and will have no practical effect for many centuries to come. Now at 1850 Jan. 0, Greenwich mean noon,

$$T=t=0, \quad \tau_0=A_0=L_0$$

and the effect of adding one mean day to T or t is

$$24^{\text{h}}=360^\circ=(\omega-\mu)/365\cdot25$$

whence

$$\omega/365\cdot25=24^{\text{h}}+(L_1-m_1)/365\cdot25$$

$$(\omega+m_1)/365\cdot25=24^{\text{h}}+L_1/365\cdot25.$$

Now, according to Newcomb,

$$L_0=279^\circ\,47'\,58''\cdot2=18^{\text{h}}\,39^{\text{m}}\,11^{\text{s}}\cdot88$$

$$L_1=1296027''\cdot6674=86401^{\text{s}}\cdot84449$$

$$L_2=+\,0''\cdot000\,1089=+\,0^{\text{s}}\cdot000\,00726$$

while in the latter unit ($1^{\text{s}}=15''$)

$$m_1=+\,3^{\text{s}}\cdot07141, \quad m_2=+\,0^{\text{s}}\cdot000\,00928$$

so that

$$L_1/365\cdot25=236^{\text{s}}\cdot55533, \quad (L_1-m_1)/365\cdot25=236^{\text{s}}\cdot54692.$$

Hence in numbers the equation (12) for Gr. sidereal time becomes

$$\tau=18^{\text{h}}\,39^{\text{m}}\,11^{\text{s}}\cdot88+(24^{\text{h}}\,3^{\text{m}}\,56^{\text{s}}\cdot55533)\,D+0^{\text{s}}\cdot000\,00928\,t^2-\Phi\cos\epsilon_0$$

where $D=365\cdot25\,t$ is the number of days reckoned from 1850 Jan. 0. When D is given an integral value this expression gives the sidereal time at Gr. mean noon and its value (less a multiple of 24^{h}) is tabulated for every day in the *Nautical Almanac.* When the nutational term is omitted,

$$\Delta\tau=(24^{\text{h}}\,3^{\text{m}}\,56^{\text{s}}\cdot55533+0^{\text{s}}\cdot000\,00005\,t)\,\Delta D.$$

The secular term is also negligible, and hence

$$\frac{1 \text{ mean day}}{1 \text{ sidereal day}}=\frac{86636^{\text{s}}\cdot555}{86400^{\text{s}}}=1\cdot002\,7379.$$

Another period which differs little from the sidereal day, but must not be confounded with it, is the period of the Earth's rotation on its axis, measured by ω. Its ratio to the mean sidereal day is

$$\frac{\omega + m_1}{\omega} = \frac{86636{\cdot}555}{86636{\cdot}547} = 1{\cdot}000\,000\,097.$$

272. A catalogue of astronomical positions gives mean places freed from nutation and reduced to the equinox of a common epoch. Such an epoch is always the beginning of a tropical year and this expression must be defined. It is the moment when the mean longitude of the Sun as above described,

$$L = L_0 + L_1 t + L_2 t^2$$

is $280^\circ = 18^h\,40^m$. It follows that the length of a tropical year is

$$\frac{24^h}{L_1 + 2L_2 t} \cdot 365{\cdot}25 \text{ mean days}$$

$$= \frac{365{\cdot}25}{1{\cdot}000\,021\,3483 + 0{\cdot}000\,000\,000\,168\,t}$$

$$= 365{\cdot}242\,20272 - 0{\cdot}000\,000\,0614\,t$$

or $365{\cdot}242200$ mean solar days at the epoch 1900. For the present the secular change is unimportant. Once the beginning of the tropical year is fixed in a particular calendar year, its beginning in any other year may be found by adding so many tropical years. But the details will be better illustrated by a direct example from the year 1900. When $t = 50$, $L' = 18^h\,40^m\,44^s{\cdot}123$. Now 50 Julian years exceed 50 years of 365 days by $12\frac{1}{2}$ days, whereas the calendar inserts 12 leap days between 1850 and 1900. Hence this is the mean longitude for 1900 Jan. 0·5. The mean longitude for 1900 Jan. 0 (Gr. mean noon) is therefore $L' - \frac{1}{2}L_1/365{\cdot}25 = 18^h\,38^m\,45^s{\cdot}845$ and must be increased by $74^s{\cdot}155$ at the daily rate $236^s{\cdot}555$ in order to become $18^h\,40^m$. This requires 0·3135 mean days, and the beginning of the tropical year in 1900 is therefore Jan. 0·3135, the fraction of a mean day being reckoned from Greenwich mean noon. This epoch is recorded briefly as 1900·0. It is to the mean equinox of this date that the observations of the year are reduced in the first instance.

273. Such in outline are the main features in the astronomical methods of reckoning time. They involve certain constants which, being based on the comparison of theory with observations, are capable of improvement. But there is no absolute standard of time. Ultimately no doubt the continued comparison of theory with observation according to such a system of time as that described above will bring to light discrepancies in the motions of the heavenly bodies of a kind which cannot be attributed to errors of

observation. Then the question will arise whether these discrepancies can be removed by a mere adjustment of an accepted system of constants involved in the measure of time or whether the fault lies in the theory. This is the ordinary experience of practical astronomy. It may, however, prove that what have been regarded as constants are not really constant at all. Thus ω, the rate of rotation of the Earth on its axis, may vary owing to such causes as the secular cooling of the Earth and the effect of tidal friction. There is, indeed, reason to think that this is so. But ultimately it is only possible to adopt such a system of measuring time as will reconcile all celestial phenomena as far as may be with the simplest possible body of laws. In the meantime to deal with discrepancies as they arise is among the most critical problems of technical astronomy.

CHAPTER XXIII

LIBRATION OF THE MOON

274. The form of solution found suitable in discussing the rotation of the Earth depends on special circumstances and is by no means general. The Moon's rotation similarly presents quite special features which require very different treatment. This movement is governed to a high degree of approximation by Cassini's laws:

(1) The Moon rotates uniformly about an axis which is fixed with respect to the Moon itself. The period of this rotation is identical with the sidereal period of the Moon in its orbit, namely 27·321661 days.

(2) The pole of the lunar rotation z makes a constant angle ($1^\circ\,35'$) with the pole of the ecliptic Z, which may here be regarded as a fixed point on the celestial sphere.

(3) In consequence of the nearly uniform regression of the lunar node on the plane of the ecliptic and the nearly constant inclination of the lunar orbit ($5^\circ\,9'$), the pole of the Moon's orbit P is known to describe a small circle about Z in a period of $18\frac{2}{3}$ years. The arc of a great circle zP contains also the pole Z. In other words, the planes of the lunar orbit and the lunar equator intersect on the ecliptic, the latter plane being intermediate between the two former.

These laws were discovered by observation and they are so exact that later work with more refined instruments has failed hitherto to determine any divergences from them with a satisfactory degree of certainty. They define as it were a steady state of motion, and it is necessary to inquire under what conditions such a state is possible, and to what oscillations it is subject according to theory.

275. The first of the above laws corresponds with the well-known fact that the Moon always presents the same face to the Earth, or more truly that a large fraction of its surface (nearly $\frac{3}{7}$) is always concealed from observation. In order that exactly the same face should be seen at all times three further conditions would be necessary and the failure of these conditions gives rise to three distinct components of what is called the apparent or

optical libration of the Moon. These conditions and the corresponding effects of their departure from the facts are:

(1) The motion of the Moon in its orbit about the Earth must be uniform. But owing to the equation of the centre and periodic perturbations the actual place of the Moon may differ from its mean place by as much as 8°. Hence an oscillation in the central meridian, which is known as the *libration in longitude.*

(2) The axis of the Moon must be normal to the plane of its orbit. Actually the angle which it makes with the normal to the orbit is

$$1^\circ\,35' + 5^\circ\,9' = 6^\circ\,44'.$$

The monthly effect of this is called the *libration in latitude.*

(3) The point of observation must be the centre of the Earth. Owing to the position of the observer on the Earth's surface, which varies with the rotation of the Earth, there is a parallactic effect which is called the *diurnal libration.*

These three effects which together constitute the optical libration of the Moon are purely geometrical consequences of the known conditions, and entirely independent of the dynamical libration which is now to be examined.

276. When the rotation of the Moon is in question the action of the Earth as a disturbing body is clearly preponderant and the action of the Sun is neglected. Let O be the centre of gravity of the Moon, $OXYZ$ a set of ecliptic axes, fixed in space, and $Oxyz$ a set fixed in the rotating body and coinciding with the principal axes of the Moon, the corresponding moments of inertia being A, B, C. Now since the axis of rotation is nearly or quite fixed in the body it must practically coincide with a principal axis; for a permanent axis in any other position would require a constraint which is obviously absent in this case. This principal axis will be identified with Oz. As in § 255 the two sets of axes are connected by the angles θ, ϕ and ψ, and $\theta = ZOz$ being always of the order $1^\circ\!\cdot\!6$, its square may be neglected. The relations between the coordinates are then given by the scheme:

	X	Y	Z
x	$\cos(\phi+\psi)$	$\sin(\phi+\psi)$	$-\theta\cos\psi$
y	$-\sin(\phi+\psi)$	$\cos(\phi+\psi)$	$\theta\sin\psi$
z	$\theta\cos\phi$	$\theta\sin\phi$	1

and Euler's geometrical equations become

$$\omega_1 = \dot\theta\sin\psi - \dot\phi\theta\cos\psi$$
$$\omega_2 = \dot\theta\cos\psi + \dot\phi\theta\sin\psi$$
$$\omega_3 = \dot\psi + \dot\phi.$$

The dynamical equations are again of the form

$$A\dot{\omega}_1 - (B - C)\,\omega_2\omega_3 = L$$
$$B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 = M$$
$$C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 = N$$

where (§ 257)

$$L = 3Gm\,(C - B)\,yz/r^5, \quad M = 3Gm\,(A - C)\,xz/r^5, \quad N = 3Gm\,(B - A)\,xy/r^5$$

m being the mass of the Earth, (x, y, z) its coordinates and r its distance from the Moon. Let (X, Y, Z) be the ecliptic coordinates of the Earth relative to the Moon. The inclination of the Moon's orbit, $c = 5^\circ\, 9'$, is so small that c^2 will be neglected. Then (cf. § 65)

$$-X = r\cos(\Omega + \omega + w), \quad -Y = r\sin(\Omega + \omega + w), \quad -Z = rc\sin(\omega + w)$$

where Ω is the longitude of the Moon's node, $(\Omega + \omega)$ the longitude of the Moon's perigee, and w the Moon's true anomaly. But

$$\lambda = \Omega + \omega + w$$

is the longitude of the Moon in its orbit. Hence, by the above relations between the two sets of coordinates,

$$-x = r\cos(\lambda - \phi - \psi), \quad -y = r\sin(\lambda - \phi - \psi)$$
$$-z = r\theta\cos(\lambda - \phi) + rc\sin(\lambda - \Omega)$$

the product $c\theta$ being neglected in x and y. Let

$$C - B = A\alpha, \quad A - C = B\beta, \quad B - A = C\gamma.$$

Then the dynamical equations of motion become

$$\left.\begin{aligned}\dot{\omega}_1 + \alpha\omega_2\omega_3 &= 3Gm\alpha r^{-3}\sin(\lambda - \phi - \psi)\,\{\theta\cos(\lambda - \phi) + c\sin(\lambda - \Omega)\}\\ \dot{\omega}_2 + \beta\omega_3\omega_1 &= 3Gm\beta r^{-3}\cos(\lambda - \phi - \psi)\,\{\theta\cos(\lambda - \phi) + c\sin(\lambda - \Omega)\}\\ \dot{\omega}_3 + \gamma\omega_1\omega_2 &= \tfrac{3}{2}Gm\gamma r^{-3}\sin 2(\lambda - \phi - \psi)\end{aligned}\right\}\ldots(1)$$

As the figure of the Moon is to all appearance sensibly spherical, α, β and γ must be fairly small quantities. And since, further, the instantaneous axis is nearly fixed in the body and very close to the axis of z, ω_1 and ω_2 must be very small in comparison with ω_3.

277. It follows that in the last equation the term $\gamma\omega_1\omega_2$ can be neglected. Hence this equation becomes, in view of the third geometrical equation,

$$\ddot{\phi} + \ddot{\psi} = \tfrac{3}{2}Gm\gamma r^{-3}\sin 2(\lambda - \phi - \psi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The Moon's mean longitude is $n't + \epsilon$, where n' is the Moon's mean motion and ϵ is a constant. The Earth's mean longitude, as seen from the Moon, is therefore $\pi + n't + \epsilon$. But according to Cassini's first law,

$$\omega_3 = \dot{\phi} + \dot{\psi} = n'$$

or

$$\phi + \psi = n't + \text{const.}$$

the constant depending on the choice of a fixed meridian on the Moon's surface. Let it be so chosen that the latter expression is equal to the Earth's mean longitude. The corresponding meridian is called the *first lunar meridian*. In order now to allow for a possible inequality in the Moon's rotation an angle χ is introduced such that

$$\phi + \psi + \chi = \pi + n't + \epsilon \quad \text{.........................}(3)$$

This angle represents an oscillation in the position of the first meridian. According to Cassini's laws $\chi = 0$ and observation proves that χ is certainly very small. The equation (2) now becomes

$$\ddot{\chi} = -\tfrac{3}{2} Gm\gamma r^{-3} \sin 2(\chi + \lambda - n't - \epsilon) \quad \text{..................}(4)$$

It is clear that the conditions of stability are only complicated by the inequalities in the motion of the Moon. Therefore we substitute for the moment a uniform circular orbit with mean distance a', so that $\lambda = n't + \epsilon$, $r = a'$ and

$$\begin{aligned} \ddot{\chi} &= -\tfrac{3}{2} Gm\gamma a'^{-3} \sin 2\chi \\ &= -\tfrac{3}{2} n'^2 \gamma (1 + f)^{-1} \sin 2\chi \quad \text{......................}(5) \end{aligned}$$

where f is the ratio of the mass of the Moon to the mass of the Earth; since by Kepler's third law

$$Gm(1 + f) = n'^2 a'^3 \quad \text{............................}(6)$$

But the equation of motion of a simple pendulum of length l and inclined to the vertical at an angle θ is

$$\ddot{\theta} = -gl^{-1} \sin \theta$$

which can be identified with (5) by taking $\chi = \tfrac{1}{2}\theta$ and $3n'^2\gamma(1 + f)^{-1} = gl^{-1}$. Both equations can of course be solved generally in elliptic integrals. But it is enough to notice the physical fact that the pendulum is capable of small vibrations provided θ is small initially and g is positive. Similarly χ if initially small will remain small provided γ is positive, i.e. $B > A$. Now, if the inclination of the lunar equator to the lunar orbit be neglected, $(\phi + \psi)$ measures the displacement of the axis of x from the equinox from which the longitudes are reckoned. Under these simplified conditions the first meridian contains the axis of x and always coincides with the central meridian of the apparent disc. The axis of x is therefore directed approximately towards the Earth and this defines the axis about which the moment A is less than the moment B. This is the first condition of stability. It is also to be inferred that $A \neq B$. For if $A = B$, $\ddot{\chi} = 0$ and a small disturbance would introduce a secular term in χ which observation shows to be absent.

278. If $\gamma' = \gamma(1 + f)^{-1}$ the more general equation (4) for χ becomes

$$\ddot{\chi} = -\tfrac{3}{2} n'^2 \gamma' (a'/r)^3 \sin 2(\chi + \lambda - n't - \epsilon).$$

Now $(\lambda - n't - \epsilon)$ is of the order of the eccentricity of the lunar orbit (·055), χ is still smaller and a'/r differs from 1 also by a quantity of the

order of the eccentricity. Hence if the square of the eccentricity be neglected,

$$\ddot{\chi} = -3n'^2\gamma'(\chi + \lambda - n't - \epsilon)$$

or

$$\ddot{\chi} + 3n'^2\gamma'\chi = -3n'^2\gamma'\Sigma H \sin(ht + h')$$

where the terms under Σ represent the equation of the centre and periodic inequalities of the lunar motion. This is the ordinary equation for forced vibrations and the solution may be written in the form $\chi = \chi_1 + \chi_2$ where χ_1 is a particular solution, corresponding to the forced vibrations, and χ_2 is the complementary function, corresponding to an arbitrary free vibration. It is easily verified that

$$\chi_1 = 3n'^2\gamma'\Sigma \frac{H}{h^2 - 3n'^2\gamma'} \sin(ht + h')$$

and

$$\chi_2 = K \sin[n't\sqrt{(3\gamma')} + k']$$

where K, k' are arbitrary. Terms in χ_1 can only become sensible by reason of H large or h small, and the most promising terms in the lunar theory are consequently the equation of the centre (or principal elliptic term):

$$ht + h' = g_1, \quad H = +22639''{\cdot}1, \quad h = 47033''{\cdot}97$$

and the annual equation:

$$ht + h' = \odot, \quad H = -668''{\cdot}9, \quad h = 3548''{\cdot}16$$

where g_1 is the Moon's mean anomaly, $\odot$ is the Sun's mean anomaly, and the unit of time is the mean solar day, so that $n' = 47435''{\cdot}03$. The corresponding terms in χ_1 are

$$\chi_1 = \frac{377'}{0{\cdot}3277 - \gamma'} \cdot \gamma' \sin g_1 - \frac{11'{\cdot}15}{0{\cdot}001865 - \gamma'} \cdot \gamma' \sin \odot \quad \text{.........(7)}$$

It is easily seen that, γ' being certainly very small, it is the second of these terms which is the larger. But the determination of its coefficient from observation has not yet been made with satisfactory certainty. Since the Earth's distance is about 220 times the Moon's radius a geocentric angle of 1″ is the equivalent of 4′ in selenographic arc near the centre of the lunar disc. As the quantities to be looked for are likely to be of this order, or rather still less, and the observations are very difficult, positive results must be awaited from the study of the large-scale photographs of the Moon which are now available. According to Franz, using the heliometer observations of Schlüter, the coefficient of sin $\odot$ is about 2′, giving γ of the order 0·0003, and the arbitrary libration K, which should have a period of rather more than 2 years, is practically negligible.

279. Since, by (3), $\omega_3 + \dot{\chi} = n'$ where $\dot{\chi}$ may now be supposed very small, the first two dynamical equations may be written

$$\left.\begin{aligned} \dot{\omega}_1 + \alpha n'\omega_2 &= L/A \\ \dot{\omega}_2 + \beta n'\omega_1 &= M/B \end{aligned}\right\} \quad \text{.............................(8)}$$

Now let

$$\xi = \theta \cos \psi, \quad \eta = \theta \sin \psi$$

so that

$$\left.\begin{aligned} \dot{\xi} &= \dot{\theta} \cos \psi + \dot{\phi}\theta \sin \psi - (\dot{\phi} + \dot{\psi})\, \theta \sin \psi = \omega_2 - \omega_3 \eta \\ \dot{\eta} &= \dot{\theta} \sin \psi - \dot{\phi}\theta \cos \psi + (\dot{\phi} + \dot{\psi})\, \theta \cos \psi = \omega_1 + \omega_3 \xi \end{aligned}\right\} \quad \ldots\ldots\ldots(9)$$

Again ω_3 may be replaced by n', being multiplied by ξ and η which are small. Hence (8) become

$$\ddot{\eta} - (1 - \alpha)\, n' \dot{\xi} + \alpha n'^2 \eta \ = L/A$$

$$\ddot{\xi} + (1 + \beta)\, n' \dot{\eta} - \beta n'^2 \xi = M/B.$$

Expressions for L/A, M/B have been given in (1), and if $f = 1/81$ be neglected in (6) these are

$$L/A \ = 3\alpha n'^2\ (a'/r)^3 \sin (\lambda - \phi - \psi)\, \{\theta \cos (\lambda - \phi) + c \sin (\lambda - \Omega)\}$$

$$M/B = 3\beta n'^2 (a'/r)^2 \cos (\lambda - \phi - \psi)\, \{\theta \cos (\lambda - \phi) + c \sin (\lambda - \Omega)\}$$

and as they are already of the order θ or c multiplied by α or β, the other quantities involved are only required to the first order in e, the eccentricity of the orbit. Now g_1 being the mean anomaly, by Ch. IV (9) and (30)—or in a more simple way—

$$a'/r = 1 + e \cos g_1, \quad w - g_1 = 2e \sin g_1$$

where

$$g_1 = n't + \epsilon - \varpi, \quad w = \lambda - \varpi$$

w being the true anomaly and ϖ the longitude of perigee. Also χ is insignificant here, so that by (3)

$$\phi + \psi = \pi + n't + \epsilon = g_1 + \varpi + \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Hence

$$\lambda - \phi - \psi = w - g_1 - \pi = 2e \sin g_1 - \pi$$

$$\sin (\lambda - \phi - \psi) = -\, 2e \sin g_1, \quad \cos (\lambda - \phi - \psi) = -\, 1$$

$$\left.\begin{aligned} (a'/r)^3 \sin (\lambda - \phi - \psi) &= -\, 2e \sin g_1 \\ (a'/r)^3 \cos (\lambda - \phi - \psi) &= -\, 1 - 3e \cos g_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(11)$$

Again,

$$\cos (\lambda - \phi) = -\cos (\psi + 2e \sin g_1) = -\cos \psi + 2e \sin g_1 \sin \psi$$

$$\theta \cos (\lambda - \phi) = -\, \theta \cos \psi + e\theta \cos (g_1 - \psi) - e\theta \cos (g_1 + \psi) \ldots(12)$$

and finally

$$\lambda - \Omega = w + \varpi - \Omega = g_1 + \varpi - \Omega + 2e \sin g_1$$

$$\sin (\lambda - \Omega) = \sin (g_1 + \varpi - \Omega) + 2e \sin g_1 \cos (g_1 + \varpi - \Omega)$$

$$\begin{aligned} c \sin (\lambda - \Omega) = c \sin (g_1 + \varpi - \Omega)& \\ + ce \sin (2g_1 + \varpi - \Omega) - ce \sin (\varpi - \Omega)& \ \ldots\ldots\ldots\ldots(13) \end{aligned}$$

It is now necessary to introduce (11), (12) and (13) into L/A, M/B, to reject terms of the third order in e, c and θ, and to resolve the products

of circular functions which occur into single functions. The result of this simple reduction gives

$$\left.\begin{aligned} L/A &= 3\alpha n'^2 \{ e\theta \sin(g_1 + \psi) + e\theta \sin(g_1 - \psi) - ec\cos(\varpi - \Omega) \\ &\qquad + ec\cos(2g_1 + \varpi - \Omega)\} \\ M/B &= 3\beta n'^2 \{\tfrac{5}{2} e\theta \cos(g_1 + \psi) + \tfrac{1}{2} e\theta \cos(g_1 - \psi) - \tfrac{1}{2} ec \sin(\varpi - \Omega) \\ &\qquad - \tfrac{5}{2} ec \sin(2g_1 + \varpi - \Omega) - c\sin(g_1 + \varpi - \Omega) + \theta\cos\psi\} \end{aligned}\right\} \quad \ldots(14)$$

The last term in M/B is $3\beta n'^2\xi$, which may be transferred immediately to the other side of the corresponding dynamical equation. This leaves one term only of the first order in M/B: the remaining terms in L/A and M/B are entirely of the second order.

280. Let the actual dynamical equations, after transferring the term $3\beta n'^2\xi$, be replaced by the forms

$$\left.\begin{aligned} \ddot{\eta} - (1-\alpha)\, n'\dot{\xi} + \alpha n'^2 \eta &= 3\alpha n'^2 P' \cos(pn't + q) \\ \ddot{\xi} + (1+\beta)\, n'\dot{\eta} - 4\beta n'^2 \xi &= 3\beta n'^2 P \sin(pn't + q) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(15)$$

A particular solution is $\xi = Q\sin(pn't + q)$, $\eta = Q'\cos(pn't + q)$, provided

$$\left.\begin{aligned} Q'(-p^2 + \alpha) - Q(1-\alpha)p &= 3\alpha P' \\ Q(-p^2 - 4\beta) - Q'(1+\beta)p &= 3\beta P \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

or

$$\left.\begin{aligned} \frac{Q}{\alpha(1+\beta)pP' - \beta(p^2-\alpha)P} &= \frac{Q'}{\beta(1-\alpha)pP - \alpha(p^2+4\beta)P'} \\ = \frac{3}{(p^2-\alpha)(p^2+4\beta) - (1-\alpha)(1+\beta)p^2} &= \frac{3}{\Delta} \end{aligned}\right\} \quad \ldots(17)$$

In this way any periodic terms on the right of the equations can be represented by corresponding terms in ξ and η. But the coefficients Q, Q' involve P, P' multiplied by the small quantities α or β, and are therefore extremely small unless Δ is also very small. Now $\Delta = p^2(p^2 - 1)$ when α and β are ignored and therefore, ceteris paribus, sensible terms can be obtained only when p is very near to 0 or ± 1.

Solutions of the same form constitute the complementary function and are determined by (17) when $P = P' = 0$. Then p is given by

$$\Delta = p^4 - p^2(1 - 3\beta - \alpha\beta) - 4\alpha\beta = 0$$

or

$$2p^2 = 1 - 3\beta - \alpha\beta \pm \sqrt{\{(1 - 3\beta - \alpha\beta)^2 + 16\alpha\beta\}}$$

It is enough to retain in p the terms of the first order in α, β, and thus

$$2p^2 = 1 - 3\beta - \alpha\beta \pm (1 - 3\beta - \alpha\beta + 8\alpha\beta)$$

so that if p_1, p_2 are the two roots,

$$p_1 = 1 - \tfrac{3}{2}\beta, \quad p_2 = 2\sqrt{(-\alpha\beta)}.$$

Thus the periods of the two possible terms are determined with sufficient accuracy, the former being nearly a month, and if the corresponding coefficients are Q_1, Q_1', Q_2, Q_2', then by (16) to the lowest order only

$$Q_1'/Q_1 = -1, \quad Q_2'/Q_2 = 2\sqrt{(-\beta/\alpha)}.$$

Hence a solution of (15) when 0 is substituted on the right-hand side is

$$\xi_1 = Q_1 \sin\{(1-\tfrac{3}{2}\beta)\, n't + q_1\} + Q_2 \sin\{2\sqrt{(-\alpha\beta)}\, t + q_2\}$$
$$\eta_1 = -Q_1 \cos\{(1-\tfrac{3}{2}\beta)\, n't + q_1\} + 2\sqrt{(-\beta/\alpha)}\, Q_2 \cos\{2\sqrt{(-\alpha\beta)}\, t + q_2\}$$

and as these expressions contain four arbitrary constants Q_1, Q_2, q_1, q_2 they represent the required complementary functions.

These arbitrary terms again appear to be insensible. The important point is that $\alpha\beta$ must be negative, for otherwise the circular functions would be changed into hyperbolic functions and the motion would be unstable. This means that $(C-B)(A-C)$ is negative, or again that C is *not* intermediate in magnitude between A and B. This is the second condition of stability which has been found.

281. To terms of the first order only,

$$L/A = 0, \quad M/B = -3\beta n'^2 c \sin(g_1 + \varpi - \Omega)$$

where, the secular inequality of the node being taken into account,

$$g_1 + \varpi = n't + \epsilon, \quad \Omega = \Omega_0 - \mu n't, \quad \mu = +0{\cdot}004019.$$

Thus in applying (17), $P' = 0$, $P = -c$, $p = 1 + \mu$, and therefore

$$\frac{-Q}{(1+\mu)^2 - \alpha} = \frac{Q'}{(1-\alpha)(1+\mu)} = \frac{-3\beta c}{(1+\mu)^2(2\mu+\mu^2) + (1+\mu)^2\beta(3+\alpha) - 4\alpha\beta} \quad \text{...(18)}$$

If α, β and μ be regarded as small quantities of the first order and those of the second order be neglected,

$$Q = -Q' = 3\beta c/(2\mu + 3\beta) \quad \text{.......................(19)}$$

so that ξ and η contain the terms

$$\xi_2 = \frac{3\beta c}{2\mu + 3\beta}\sin(g_1 + \varpi - \Omega), \quad \eta_2 = \frac{-3\beta c}{2\mu + 3\beta}\cos(g_1 + \varpi - \Omega) \quad \text{...(20)}$$

These terms contain the explanation of the steady motion of the Moon's axis, which is expressed by Cassini's laws.

For the coordinates of the Moon's pole of rotation relative to the pole of the ecliptic may be taken as

$$X = \theta\cos\phi = \xi\cos(\phi + \psi) + \eta\sin(\phi + \psi)$$
$$Y = \theta\sin\phi = \xi\sin(\phi + \psi) - \eta\cos(\phi + \psi).$$

Let the free components ξ_1, η_1 be ignored and also the forced oscillations of the second order which have still to be found. Then

$$X = Q\sin(g_1 + \varpi - \Omega - \phi - \psi)$$
$$Y = Q\cos(g_1 + \varpi - \Omega - \phi - \psi).$$

But by (10)

$$\phi + \psi = g_1 + \varpi + \pi$$

and therefore

$$X = Q\sin\Omega, \quad Y = -Q\cos\Omega.$$

But the longitude of the pole of the lunar orbit is $\Omega - \frac{1}{2}\pi$, so that its coordinates are similarly

$$X' = c\sin\Omega, \quad Y' = -c\cos\Omega.$$

Hence these two poles are always exactly on opposite sides of the pole of the ecliptic provided Q is negative. This requires, since Q is given by (19), $0 > \beta > -\frac{2}{3}\mu$. Hence $C > A$, which is a third condition to be satisfied by the moments of inertia. The resultant of the three places the moments in the order

$$C > B > A$$

where C refers to the axis of rotation and A to that axis which in the mean is directed towards the Earth.

It is now clear that the further conditions necessary in order that the second and third laws of Cassini shall remain approximately true are one and the same, namely that those terms which have been neglected in the above argument are really small in comparison with Q. This quantity is the mean value of θ, and its numerical value is $91'\cdot4$ according to Franz. With $c = 308'\cdot7$ and $\mu = 0\cdot004019$ it follows that

$$-\beta = (C - A)/B = 0\cdot000612$$

which should be tolerably well determined. It is to be noticed that α, β, γ are not independent, but connected by the identity

$$\alpha + \beta + \gamma + \alpha\beta\gamma = 0.$$

The product is negligible and if $\gamma = 0\cdot0003$ as given above, then α is of exactly the same order as γ.

282. The terms of the second order in e, c, θ can now be found without difficulty, since here it is legitimate to give θ and ψ their values in the steady motion. Thus $\theta = \theta_0$, its constant mean value, and since in the steady motion $\phi = \Omega + \frac{1}{2}\pi$,

$$\psi = g_1 + \varpi - \Omega + \tfrac{1}{2}\pi.$$

Hence without the terms of lower order already treated, the expressions (14) become

$$L/A = 3\alpha n'^2\{e(\theta_0 + c)\cos(2g_1 + \varpi - \Omega) - e(\theta_0 + c)\cos(\varpi - \Omega)\}$$
$$M/B = 3\beta n'^2\{-\tfrac{3}{2}e(\theta_0 + c)\sin(2g_1 + \varpi - \Omega) - \tfrac{1}{2}e(\theta_0 + c)\sin(\varpi - \Omega)\}.$$

The corresponding terms in ξ, η can be found in the way explained in § 280. But since ϖ and Ω change slowly p is nearly 2 in the case of the terms which contain $2g_1$ in the argument. Their counterpart in ξ, η is therefore negligible. With the other pair p is very small. The secular changes in the node and perigee may be expressed by

$$\Omega = \Omega_0 - \mu n't, \quad \varpi = \varpi_0 + \nu n't$$

so that $p = \mu + \nu$, and $2P = P' = -e(\theta_0 + c)$. Hence (17) give

$$\frac{Q}{2\alpha(1+\beta)p - \beta(p^2-\alpha)} = \frac{Q'}{\beta(1-\alpha)p - 2\alpha(p^2+4\beta)}$$
$$= \frac{-\frac{3}{2}e(\theta_0+c)}{(p^2-\alpha)(p^2+4\beta)-(1-\alpha)(1+\beta)p^2}$$

which, when simplified by the removal of all but the most significant quantities in the denominators, become

$$Q/2\alpha = Q'/\beta = \tfrac{3}{2}e(\theta_0+c)/p.$$

The terms of the second order are therefore simply

$$\xi_3 = 3\alpha e\frac{\theta_0+c}{\mu+\nu}\sin(\varpi-\Omega), \quad \eta_3 = \tfrac{3}{2}\beta e\frac{\theta_0+c}{\mu+\nu}\cos(\varpi-\Omega) \quad \text{......(21)}$$

Now $\nu = 0{\cdot}008455$, $\mu + \nu = 1/80$ nearly, and $\theta_0 + c = 400'$. Also $e = 0{\cdot}0549$ and with the above values of α and β, $3\alpha e = -\frac{3}{2}\beta e = 0{\cdot}00005$. Hence both coefficients are numerically $1''{\cdot}6$, and

$$\xi_3 = 1''{\cdot}6\sin(\varpi-\Omega), \quad \eta_3 = -1''{\cdot}6\cos(\varpi-\Omega)$$

the period being 80 lunar months or 6 years.

283. When the several terms found are combined,

$$\xi = \xi_1 + \xi_2 + \xi_3, \quad \eta = \eta_1 + \eta_2 + \eta_3$$

and by (9)

$$\omega_1 = \dot\eta - \omega_3\xi, \quad \omega_2 = \dot\xi + \omega_3\eta.$$

Now with the approximate forms (20)

$$\dot\xi_2 = -n'\eta_2, \quad \dot\eta_2 = n'\xi_2$$

and from (21)

$$\dot\xi_3 = n'(\mu+\nu)\eta_3, \quad \dot\eta_3 = -n'(\mu+\nu)\xi_3.$$

Hence, putting $\omega_3 = n'$ here and neglecting the arbitrary terms ξ_1, η_1, the existence of which has not been established by observation,

$$\omega_1/n' = -(1+\mu+\nu)\xi_3, \quad \omega_2/n' = (1+\mu+\nu)\eta_3$$

and $(\mu + \nu)$ is relatively unimportant here.

One remark is necessary however. For the sake of simplicity and in order to concentrate attention on the main feature of the motion, the coefficients of ξ_2 and η_2 in (20) were made numerically equal by the simple expedient of neglecting $\mu^2 (= 0{\cdot}000016)$ in comparison with μ. Consistently with this

the factor $(1+\mu)$ has been omitted in finding $\dot{\xi}_2$, $\dot{\eta}_2$, and the result is that ξ_2, η_2 do not appear in ω_1, ω_2. This factor can only be reinstated correctly after μ^2 has been restored in ξ_2, η_2. Now by (18) ξ_2, η_2 are of the form

$$\xi_2 = \{(1+\mu)^2 - \alpha\}\, G \sin g, \quad \eta_2 = -(1-\alpha)(1+\mu)\, G \cos g$$

where $g = g_1 + \varpi - \Omega$. Hence

$$\dot{\xi}_2/n' = (1+\mu)\{(1+\mu)^2 - \alpha\}\, G \cos g$$

$$\dot{\eta}_2/n' = (1+\mu)^2 (1-\alpha)\, G \sin g$$

and the contributions to ω_1, ω_2 are given by

$$\Delta\omega_1/n' = -\alpha(2\mu+\mu^2)\, G \sin g$$

$$\Delta\omega_2/n' = (1+\mu)(2\mu+\mu^2)\, G \cos g.$$

The factor α shows that $\Delta\omega_1$ is very small and if μ^2 as well as α be now rejected,

$$\Delta\omega_1/n' = 0, \quad \Delta\omega_2/n' = -2\mu\eta_2.$$

Hence in a numerical form the forced rotations are finally given by

$$\omega_1/n' = -\xi_3 = -1''\cdot 6 \sin(\varpi - \Omega)$$

$$\omega_2/n' = \eta_3 - 2\mu\eta_2 = -1''\cdot 6 \cos(\varpi - \Omega) - 0''\cdot 7 \cos(g_1 + \varpi - \Omega)$$

since $G = -91''\cdot 4$ and $\mu = 0\cdot 004$.

With the more exact expressions the coefficient in ξ_2 is numerically greater than that in η_2, the difference being $-\mu(1+\mu+\alpha)\,G$ or $-\mu G$. This amount, $22''$, may be divided equally between the two coefficients without disturbing the observed mean inclination of the lunar equator to the lunar orbit, and thus

$$\xi_2 = -91''\cdot 6 \sin(g_1 + \varpi - \Omega), \quad \eta_2 = 91''\cdot 2 \cos(g_1 + \varpi - \Omega).$$

Lastly, by (7), if χ_2 the free libration in longitude be ignored,

$$\omega_3/n' = 1 - \dot{\chi}n' = 1 - \frac{0\cdot 11}{0\cdot 33 - \gamma'} \cdot \gamma' \cos g_1 + \frac{0\cdot 000242}{0\cdot 001865 - \gamma'} \cdot \gamma' \cos \odot$$

where the coefficients are expressed in circular measure. Thus the position of the instantaneous axis, relative to the principal axes of the Moon,

$$x/\omega_1 = y/\omega_2 = z/\omega_3$$

is determined. It has therefore been seen under what conditions Cassini's laws are approximately true, and how far they must necessarily be modified by disturbing actions.

The latest results from observation, by M. Puiseux of Paris, seem to be at variance with the foregoing theory. It is probable that it will be necessary to treat the Moon as a deformable body, as the observed variations of latitude have shown to be requisite in the case of the Earth. The above theory is very largely due to Poisson.

CHAPTER XXIV

FORMULAE OF NUMERICAL CALCULATION

284. If we consider a function of one variable or *argument* only, for the sake of definiteness, it can be represented in three distinct ways, namely:

(1) By an analytical form, e.g. $\sin x$ or a hypergeometric series $F(\alpha, \beta, \gamma, x)$. The effectiveness of such a form depends on the knowledge of its properties and the facility with which it submits to the ordinary operations of mathematics.

(2) Graphically, by a curve. This gives a continuous representation. Values of the function corresponding to particular values of the argument can be obtained and the processes of differentiation and integration can be performed mechanically. But the accuracy of the results is limited in practice.

(3) Numerically, by a series of isolated values. This gives a discontinuous representation, but one capable of very great accuracy. In theory this does not serve to define the function, for it may vary in any manner between the given values. Even in practice the representation does not cover terms in the function with a period of the same order as the intervals between the values. But with due care this limitation causes little inconvenience.

Each mode of representation has distinct advantages of its own and to pass from one to another is a problem frequently arising and often attended by great difficulty. The form (1) may be considered the ultimate expression of natural truth, but it has no absolute superiority. Thus integration may be practically impossible in this form and must be replaced by a mechanical quadrature.

A function determined by a series of observations or experiments falls generally under the form (3). Now the variable quantities which occur in Astronomy, e.g. the coordinates of the Moon, are in general so complicated, even when an expression in analytical form is available, that for practical purposes it is necessary to use an *ephemeris*, or a table of values calculated for equal intervals of time (not necessarily one day, as the name would imply). It is therefore necessary to consider how functions represented in

this way may be manipulated so as to give intermediate values by interpolation for comparison with the results of observation, and also to render numerical differentiation and integration possible.

285. Let w be the constant interval of the argument and $y_n = f(a + nw)$ be the function to be considered, the values of y_n being given for consecutive integral values of n. A simple difference table can be formed thus:

$$\begin{array}{c|ccc}
a+(n-1)w & y_{n-1} & & \cdots \\
 & & y_n - y_{n-1} & \\
a+nw & y_n & & y_{n+1} - 2y_n + y_{n-1} \\
 & & y_{n+1} - y_n & \\
a+(n+1)w & y_{n+1} & & \cdots
\end{array}$$

Now let two operators Δ, δ be introduced such that

$$\Delta y_n = y_{n+1} - y_n, \quad \delta y_n = y_n - y_{n-1}.$$

Then it follows that

$$\Delta\delta y_n = \Delta(y_n - y_{n-1}) = y_{n+1} - 2y_n + y_{n-1} = \delta(y_{n+1} - y_n) = \delta\Delta y_n.$$

Hence the operators Δ, δ are commutative, and similarly it is easily seen that they obey all the laws of ordinary algebra. The inverse operators Δ^{-1}, δ^{-1} may be defined so that $\Delta\Delta^{-1} = 1$, $\delta\delta^{-1} = 1$. Then the table of differences may be replaced by a table of operations which, acting on y_n, will reproduce the difference table, thus:

$$\begin{array}{|ccc}
\Delta^{-1}\delta & & \delta^2 \\
 & \delta & \\
1 & & \Delta\delta \\
 & \Delta & \\
\Delta\delta^{-1} & & \Delta^2
\end{array}$$

The two operators are not independent, for the position of $\Delta\delta$ in this table shows that they are connected by the homographic relation

$$\Delta\delta = \Delta - \delta, \quad \delta = \Delta(1+\Delta)^{-1}, \quad \Delta = \delta(1-\delta)^{-1} \quad \text{............(1)}$$

Let x be the variable, so that $y = f(x)$, and let $D = d/dx$. Then

$$\begin{aligned}
(1+\Delta)f(x) &= f(x+w) \\
&= f(x) + wf'(x) + \tfrac{1}{2}w^2 f''(x) + \ldots \\
&= \{1 + wD + \tfrac{1}{2}w^2D^2 + \ldots\} f(x) \\
&= e^{wD} . f(x) \quad \text{..(2)}
\end{aligned}$$

or $1 + \Delta = e^{wD}$. Hence

$$\begin{aligned}
(1+\Delta)^q f(x) &= e^{qwD} . f(x) \\
&= f(x) + qwf'(x) + \tfrac{1}{2}q^2w^2 f''(x) + \ldots \\
&= f(x+qw).
\end{aligned}$$

Thus

$$f(x+qw) = \left\{1 + q\Delta + \binom{q}{2}\Delta^2 + \ldots\right\} f(x)$$

which is Newton's original formula of interpolation and can be written in the form:

$$y_{n+q} = \left\{1 + q\Delta + \binom{q}{2}\Delta^2 + \ldots\right\} y_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $|q|$ by a proper choice of n may always be taken $< \frac{1}{2}$, and in any case should not exceed 1. The coefficients are simple binomial coefficients.

286. The differences Δ, $\Delta^2, \ldots$ are diagonal differences in the table. But the most useful formulae involve *central* differences, lying on or adjacent to a horizontal line in the table. If the blank spaces in the odd columns are filled by the arithmetic means of the entries immediately above and below, the operators in the complete central line are

$$1 \qquad \tfrac{1}{2}(\Delta + \delta) \qquad \Delta\delta \qquad \tfrac{1}{2}(\Delta + \delta)\,\Delta\delta \qquad (\Delta\delta)^2 \quad \ldots$$

which can also be written, by introducing two new operators K, k,

$$1 \qquad k \qquad K \qquad kK \qquad K^2 \quad \ldots$$

where

$$\left.\begin{aligned} k = \tfrac{1}{2}(\Delta + \delta), \quad K = \Delta\delta = \Delta - \delta \\ \Delta = k + \tfrac{1}{2}K, \quad \delta = k - \tfrac{1}{2}K, \quad k^2 - \tfrac{1}{4}K^2 = K \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Thus k cannot be expressed rationally in terms of K, and in order to find a formula in terms of central differences it is necessary to expand in terms of K, keeping only the first power of k. Thus

$$(1 + \Delta)^q = (1 + k + \tfrac{1}{2}K)^q = ku_q + v_q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where

$$u_q = \binom{q}{1}(1 + \tfrac{1}{2}K)^{q-1} + \binom{q}{3}(1 + \tfrac{1}{2}K)^{q-3}(K + \tfrac{1}{4}K^2) + \ldots$$

$$v_q = (1 + \tfrac{1}{2}K)^q + \binom{q}{2}(1 + \tfrac{1}{2}K)^{q-2}(K + \tfrac{1}{4}K^2) + \ldots.$$

It is easily verified that

$$u_q(1 + \tfrac{1}{2}K) + v_q = u_{q+1}, \quad u_q(K + \tfrac{1}{4}K^2) + v_q(1 + \tfrac{1}{2}K) = v_{q+1}$$

since

$$\binom{q}{r} + \binom{q}{r-1} = \binom{q+1}{r}.$$

Also

$$\frac{dv_q}{dK} = \sum_r \binom{q}{2r}\{\tfrac{1}{2}(q - 2r)(1 + \tfrac{1}{2}K)^{q-2r-1}(K + \tfrac{1}{4}K^2)^r + r(1 + \tfrac{1}{2}K)^{q-2r+1}(K + \tfrac{1}{4}K^2)^{r-1}\}$$

$$= \Sigma\left\{\tfrac{1}{2}(q - 2r)\binom{q}{2r} + (r + 1)\binom{q}{2r+2}\right\}(1 + \tfrac{1}{2}K)^{q-2r-1}(K + \tfrac{1}{4}K^2)^r$$

$$= \Sigma\,\tfrac{1}{2}q\left\{\binom{q-1}{2r} + \binom{q-1}{2r+1}\right\}(1 + \tfrac{1}{2}K)^{q-2r-1}(K + \tfrac{1}{4}K^2)^r$$

$$= \tfrac{1}{2}q\Sigma\binom{q}{2r+1}(1 + \tfrac{1}{2}K)^{q-2r-1}(K + \tfrac{1}{4}K^2)^r = \tfrac{1}{2}qu_q.$$

It is therefore possible to write

$$v_q = 1 + q\Sigma b_r K^r, \quad u_q = q + 2\Sigma (r+1)\, b_{r+1} K^r.$$

Let b_r become b_r' in v_{q+1}, u_{q+1}, and equate the coefficients of K^{r-1} in the first, and of K^r in the second, recurrence formula. Thus

$$2rb_r' = 2rb_r + (r-1)\, b_{r-1} + qb_{r-1}$$

$$(q+1)\, b_r' = 2rb_r + \tfrac{1}{2}(r-1)\, b_{r-1} + qb_r + \tfrac{1}{2}qb_{r-1}$$

and, on eliminating b_r',

$$2r(2r-1)\, b_r = (q+r-1)(q-r+1)\, b_{r-1}.$$

This shows that

$$b_r = \binom{q+r-1}{2r-1} \frac{A}{2r}$$

where A is a constant, and since $b_1 = \tfrac{1}{2}q$, $A = 1$. Hence

$$u_q = q + \Sigma \binom{q+r}{2r+1} K^r, \quad v_q = 1 + q\Sigma \binom{q+r-1}{2r-1} \frac{K^r}{2r} \quad \text{.........(6)}$$

and the first terms of the complete formula are therefore

$$y_{n+q} = \left\{1 + q \,.\, k + \frac{q^2}{2\,!} \,.\, K + \frac{q(q^2-1^2)}{3\,!} \,.\, kK + \frac{q^2(q^2-1^2)}{4\,!} \,.\, K^2 \right.$$
$$\left. + \frac{q(q^2-1^2)(q^2-2^2)}{5\,!} \,.\, kK^2 + \ldots \right\} y_n \quad \text{......(7)}$$

This series was found by Newton, but is generally known as Stirling's formula. It is here taken as fundamental, and other results are deduced from it.

287. The formula of Gauss depends on the even central differences and the odd differences of the line below, the operators being therefore

$$\begin{array}{ccccc} 1 & & K & & K^2 \\ & \Delta & & \Delta K & & \ldots. \end{array}$$

These are, in terms of k, K,

$$1, \quad k + \tfrac{1}{2}K, \quad K, \quad (k+\tfrac{1}{2}K)\, K, \quad K^2, \quad \ldots.$$

But (5) may be written in the form

$$(1+\Delta)^q = (k + \tfrac{1}{2}K)\, u_q + (v_q - \tfrac{1}{2}Ku_q) = \Delta \,.\, u_q + V_q$$

where by (6)

$$V_q = v_q - \tfrac{1}{2}Ku_q = 1 + \Sigma \binom{q+r-1}{2r-1} \left(\frac{q}{2r} - \tfrac{1}{2}\right) K^r$$

$$= 1 + \Sigma \binom{q+r-1}{2r} K^r \quad \text{........................(8)}$$

This gives the coefficients of the even central differences, the coefficients of the odd differences of the adjacent line being still given by u_q. The first terms of the complete formula are therefore

$$y_{n+q}=\left\{1+q\,.\,\Delta+\frac{q(q-1)}{2\,!}\,.\,K+\frac{q(q^2-1^2)}{3\,!}\,.\,\Delta K+\frac{q(q^2-1^2)(q-2)}{4\,!}\,.\,K^2\right.$$
$$\left.+\frac{q(q^2-1^2)(q^2-2^2)}{5\,!}\,.\,\Delta K^2+\ldots\right\}y_n \ldots\ldots\ldots\ldots\ldots\ldots\text{(9)}$$

If the order of the difference table were reversed, $-\delta$ would take the place of Δ and the sign of w would be changed. Hence similarly

$$y_{n-q}=\left\{1-q\,.\,\delta+\frac{q(q-1)}{2\,!}\,.\,K-\frac{q(q^2-1^2)}{3\,!}\,.\,\delta K+\ldots\right\}y_n \ldots\ldots\text{(10)}$$

By choosing either (9) or (10) q can always be taken between 0 and $+\frac{1}{2}$.

288. The formula of Bessel contains the odd differences in the line immediately below the central function, with the mean even differences of the same line, so that the operators are

$$1+\tfrac{1}{2}\Delta,\quad \Delta,\quad (1+\tfrac{1}{2}\Delta)\,K,\quad \Delta K,\quad (1+\tfrac{1}{2}\Delta)\,K^2,\ \ldots.$$

The odd differences are thus the same as in the formula of Gauss, and therefore

$$(1+\Delta)^q=\Delta u_q+V_q=(1+\tfrac{1}{2}\Delta)\,V_q+\Delta\,(u_q-\tfrac{1}{2}V_q)$$
$$=(1+\tfrac{1}{2}\Delta)\,V_q+\Delta\,U_q$$

where, by (6) and (8),

$$U_q=u_q-\tfrac{1}{2}V_q=q-\tfrac{1}{2}+\Sigma\left\{\binom{q+r}{2r+1}-\tfrac{1}{2}\binom{q+r-1}{2r}\right\}K^r$$
$$=(q-\tfrac{1}{2})\left\{1+\Sigma\binom{q+r-1}{2r}\frac{K^r}{2r+1}\right\}\ \ldots\ldots\ldots\ldots\text{(11)}$$

This gives the coefficients of the odd differences, and the coefficients of the even (mean) differences are given by V_q. Hence the first terms of the complete formula are

$$y_{n+q}=\left\{(1+\tfrac{1}{2}\Delta)+(q-\tfrac{1}{2})\,.\,\Delta+\frac{q(q-1)}{2\,!}\,.\,(1+\tfrac{1}{2}\Delta)\,K+(q-\tfrac{1}{2})\,\frac{q(q-1)}{3\,!}\,\Delta K\right.$$
$$\left.+\frac{q(q^2-1^2)(q-2)}{4\,!}\,.(1+\tfrac{1}{2}\Delta)\,K^2+(q-\tfrac{1}{2})\frac{q(q^2-1^2)(q-2)}{5\,!}\,.\,\Delta K^2+\ldots\right\}y_n\ \ldots\text{(12)}$$

Bessel's own form differs from this in the first two terms, being written

$$y_{n+q}=\left\{1+q\,.\,\Delta+\frac{q(q-1)}{2\,!}\,.\,(1+\tfrac{1}{2}\Delta)\,K+\ldots\right\}y_n$$

which is of course equivalent, but is not symmetrical with respect to the middle of the tabular interval. To make this symmetry clearer, let $p+\frac{1}{2}$ be substituted for q in (12), which then becomes

$$y_{n+\frac{1}{2}+p}=\Big\{(1+\tfrac{1}{2}\Delta)+p\,.\,\Delta+\frac{p^2-\frac{1}{4}}{2\,!}\,.\,(1+\tfrac{1}{2}\Delta)\,K+p\,.\,\frac{p^2-\frac{1}{4}}{3\,!}\,.\,\Delta K$$
$$+\frac{(p^2-\frac{1}{4})(p^2-\frac{9}{4})}{4\,!}\,.\,(1+\tfrac{1}{2}\Delta)\,K^2+p\,.\,\frac{(p^2-\frac{1}{4})(p^2-\frac{9}{4})}{5\,!}\,.\,\Delta K^2+\ldots\Big\}\,y_n \quad \ldots(13)$$

When the sign of p is reversed, the terms of even order are unchanged and the terms of odd order are simply reversed in sign. If terms of the two orders are computed separately, two interpolations—corresponding to $\pm p$—are obtained at the same time. This is of great advantage in systematic interpolation to regular fractions of the tabular interval, e.g. in reducing the 12-hourly places of the Moon to an hourly ephemeris. Stirling's formula presents a similar advantage. But (13) becomes particularly simple at the middle of an interval, for then $q=\frac{1}{2}$ or $p=0$, and the odd differences disappear. Thus

$$y_{n+\frac{1}{2}}=\{(1+\tfrac{1}{2}\Delta)-\tfrac{1}{8}(1+\tfrac{1}{2}\Delta)\,K+\tfrac{3}{128}(1+\tfrac{1}{2}\Delta)\,K^2$$
$$-\tfrac{5}{1024}(1+\tfrac{1}{2}\Delta)\,K^3+\ldots\}\,y_n \quad \ldots\ldots\ldots(14)$$

and this gives intermediate values with great ease and accuracy.

289. When the values of a function y are known only at irregular intervals of the argument x, as in an ordinary series of observations, the function is strictly indeterminate in the absence of other information as to its form. Nevertheless, when n values $y_1, \ldots, y_n$ are known, corresponding to $x_1, \ldots, x_n$, a formula

$$y=a_0+a_1x+\ldots+a_{n-1}\,x^{n-1}$$

can be found which is satisfied by the n values and within the interval x_1 to x_n will generally resemble the true function closely. The n coefficients can be determined by the linear equations

$$y_r=a_0+a_1x_r+\ldots+a_{n-1}\,x_r^{\,n-1}$$

$(r=1, \ldots, n)$. These can be solved in the ordinary way, but it is immediately obvious that the result can be written

$$y=\Sigma y_r\,\frac{(x-x_1)\ldots(x-x_n)}{(x_r-x_1)\ldots(x_r-x_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

where the numerator of the fraction written does not contain $(x-x_r)$. For this equation becomes an identity when x_r, y_r are substituted for x, y. The expression on the right is a polynomial of degree $n-1$ in x and the equation, since it is satisfied by every pair (x_r, y_r), must be identical with the previous equation, the coefficients in which can be written down by comparison. The formula (15) is due to Lagrange and is directly suitable for interpolation,

differentiation and integration. An illustration of its use in a case where $n=3$ has been given in § 71. When n is large the formula naturally becomes inconvenient for practical purposes.

290. Returning to the function with known values at regular intervals of the argument, let us consider the process of mechanical differentiation. By (2)

$$\left.\begin{aligned} wD &= \log(1+\Delta) = \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \dots \\ w^2D^2 &= \{\log(1+\Delta)\}^2 = \Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \dots \end{aligned}\right\} \quad \dots\dots\dots\dots(16)$$

These formulae are suitable only in simple cases where great accuracy is not required. The loss of accuracy is a natural tendency when differentiation is concerned. The forms (16) also apply only to the tabulated value of the argument. But since

$$x = a + (n+q)\,w, \quad wD = wd/dx = d/dq$$

a formula of differentiation can be derived from every formula of interpolation. Thus Bessel's formula (12) gives

$$\left.\begin{aligned} wy'_{n+q} &= \{\Delta + \tfrac{1}{2}(2q-1)\,.\,(1+\tfrac{1}{2}\Delta)\,K + \tfrac{1}{12}(6q^2-6q+1)\,.\,\Delta K + \dots\}\,y_n \\ w^2y''_{n+q} &= \{(1+\tfrac{1}{2}\Delta)K + \tfrac{1}{2}(2q-1)\,.\,\Delta K + \tfrac{1}{12}(6q^2-6q-1)\,.\,(1+\tfrac{1}{2}\Delta)K^2 + \dots\}\,y_n \end{aligned}\right\}(17)$$

and analogous forms may be derived similarly by differentiating (7) and (9) with respect to q.

But there are some particular cases of special simplicity and importance in the formulae of central differences. According to (6) u_q is an odd function and v_q an even function of q. Now when $q=0$, d/dq is the coefficient of q and d^2/dq^2 is twice the coefficient of q^2 in $ku_q + v_q$. These coefficients can easily be taken from ku_q and v_q respectively, and give, by (6) or (7),

$$\left.\begin{aligned} wD &= k\left\{1 - \frac{1^2}{3\,!}K + \frac{1^2\,.\,2^2}{5\,!}K^2 - \frac{1^2\,.\,2^2\,.\,3^2}{7\,!}K^3 + \dots\right\} \\ wy_n' &= (k - \tfrac{1}{6}kK + \tfrac{1}{30}kK^2 - \tfrac{1}{140}kK^3 + \dots)\,y_n \end{aligned}\right\} \quad \dots\dots(18)$$

and

$$\left.\begin{aligned} \tfrac{1}{2}w^2D^2 &= \frac{1}{2\,!}K - \frac{1^2}{4\,!}K^2 + \frac{1^2\,.\,2^2}{6\,!}K^3 - \frac{1^2\,.\,2^2\,.\,3^2}{8\,!}K^4 + \dots \\ w^2y_n'' &= (K - \tfrac{1}{12}K^2 + \tfrac{1}{90}K^3 - \tfrac{1}{560}K^4 + \dots)\,y_n \end{aligned}\right\} \quad \dots\dots(19)$$

Both (18) and (19) involve the *alternate* differences in the central tabular line.

Similarly when V_q, U_q are expressed in terms of $p = q + \tfrac{1}{2}$ instead of q as in (8) and (11), V_q is an even function and U_q is an odd function of p. When $q = \tfrac{1}{2}$, $p=0$ and d/dq is the coefficient of p and d^2/dq^2 is twice the

coefficient of p^2 in $(1+\frac{1}{2}\Delta)V_q + \Delta U_q$. These coefficients can readily be taken from (13), which sufficiently indicates the law of formation, and thus

$$\left.\begin{aligned} wD(1+\Delta)^{\frac{1}{2}} &= \Delta\left\{1 - \frac{1^2}{3!}\frac{K}{4} + \frac{1^2.3^2}{5!}\left(\frac{K}{4}\right)^2 - \frac{1^2.3^2.5^2}{7!}\left(\frac{K}{4}\right)^3 + \dots\right\} \\ wy'_{n+\frac{1}{2}} &= \left\{\Delta - \tfrac{1}{6}.\frac{1}{4}\Delta k + \tfrac{3}{40}.\frac{1}{4^2}\Delta K^2 - \tfrac{5}{112}.\frac{1}{4^3}\Delta K^3 + \dots\right\} y_n \end{aligned}\right\} \dots(20)$$

and

$$\left.\begin{aligned} \tfrac{1}{2}w^2D^2(1+\Delta)^{\frac{1}{2}} &= (1+\tfrac{1}{2}\Delta)\left\{\frac{K}{2!} - (1^2+3^2)\frac{K^2}{4!4} + (3^2.5^2+1^2.5^2+1^2.3^2)\frac{K^3}{6!4^2}\right. \\ &\qquad \left. - (3^2.5^2.7^2+1^2.5^2.7^2+1^2.3^2.7^2+1^2.3^2.5^2)\frac{K^4}{8!4^3} - \dots\right\} \\ w^2y''_{n+\frac{1}{2}} &= \left\{(1+\tfrac{1}{2}\Delta)K - \tfrac{5}{6}.\frac{1}{4}(1+\tfrac{1}{2}\Delta)K^2 + \tfrac{259}{360}.\frac{1}{4^2}(1+\tfrac{1}{2}\Delta)K^3\right. \\ &\qquad \left. - \tfrac{3229}{5040}.\frac{1}{4^3}(1+\tfrac{1}{2}\Delta)K^4 + \dots\right\} y_n \end{aligned}\right\} (21)$$

The distinction between the operators $(1+\Delta)^{\frac{1}{2}}$ and $(1+\frac{1}{2}\Delta)$ must be carefully noted. That on the left, $(1+\Delta)^{\frac{1}{2}}$, indicates an addition of half the tabular interval to the argument, so as to apply the differentiation at the right point, which is the middle of the interval. That on the right, $(1+\frac{1}{2}\Delta)$, merely denotes the mean of adjacent differences in a vertical column of the difference table.

291. Convenient methods for mechanical integration or quadrature can now be deduced. The formulae for differentiation just found, (18), (19), (20), (21), are of the form

$$wD = kS_1(K), \qquad w^2D^2 = S_2(K)$$
$$wD(1+\Delta)^{\frac{1}{2}} = \Delta S_3(K), \quad w^2D^2(1+\Delta)^{\frac{1}{2}} = (1+\tfrac{1}{2}\Delta)S_4(K)$$

$S(K)$ denoting a power series in K. Hence

$$w^{-1}D^{-1} = k^{-1}/S_1(K), \qquad w^{-2}D^{-2} = 1/S_2(K)$$
$$w^{-1}D^{-1}(1+\Delta)^{\frac{1}{2}} = (1+\Delta)\Delta^{-1}/S_3(K), \quad w^{-2}D^{-2}(1+\Delta)^{\frac{1}{2}} = (1+\Delta)(1+\tfrac{1}{2}\Delta)^{-1}/S_4(K).$$

The coefficients of the reciprocals of the K series must be expressed more appropriately, thus:

$$k^{-1} = k/k^2 = k(K+\tfrac{1}{4}K^2)^{-1} = kK^{-1}/(1+\tfrac{1}{4}K)$$
$$(1+\Delta)\Delta^{-1} = \delta^{-1} = \Delta K^{-1}$$
$$(1+\Delta)(1+\tfrac{1}{2}\Delta)^{-1} = (1+\tfrac{1}{2}\Delta)\{1+\tfrac{1}{4}\Delta^2(1+\Delta)^{-1}\}^{-1} = (1+\tfrac{1}{2}\Delta)(1+\tfrac{1}{4}\Delta\delta)^{-1}$$
$$= (1+\tfrac{1}{2}\Delta)/(1+\tfrac{1}{4}K).$$

It is therefore necessary to multiply S_1 and S_4 by $(1+\frac{1}{4}K)$ before finding the reciprocals of the series by division in order to have results for D^{-1}, D^{-2} of

exactly the same form as those already found for D, D^2. These results are easily found to be

$$w^{-1} D^{-1} = k\left(K^{-1} - \tfrac{1}{12} + \tfrac{11}{720} K - \tfrac{191}{60480} K^2 +\right) \qquad (22)$$

$$w^{-2} D^{-2} = K^{-1} + \tfrac{1}{12} - \tfrac{1}{240} K + \tfrac{31}{60480} K^2 - \qquad (23)$$

$$w^{-1} D^{-1} (1+\Delta)^{\frac{1}{2}} = \Delta\left(K^{-1} + \tfrac{1}{24} - \tfrac{17}{5760} K + \tfrac{367}{967680} K^2 - \ldots\right) \qquad (24)$$

$$w^{-2} D^{-2} (1+\Delta)^{\frac{1}{2}} = (1 + \tfrac{1}{2}\Delta)\left(K^{-1} - \tfrac{1}{24} + \tfrac{17}{1920} K - \tfrac{367}{193536} K^2 + \ldots\right) \qquad (25)$$

The development is here carried as far as differences of the fifth order. This is generally sufficient.

It is now necessary to examine the meaning of these purely formal results. The operator K, like its components Δ, δ, is such that $KK^{-1} = 1$, and therefore, as K represents a move two places to the right in the table, K^{-1} represents a move two places to the left. The difference table now requires an extension not hitherto contemplated, and the central line of the table of operators, with the adjacent lines above and below, now becomes:

$$\begin{array}{cc|c|cccccc}
 & \delta K^{-1} & & \delta & & \delta K & & \delta K^2 \ldots \\
K^{-1} & [kK^{-1}] & 1 & [k] & K & [kK] & K^2 & [kK^2] \ldots \\
[(1+\frac{1}{2}\Delta) K^{-1}] & \Delta K^{-1} & [1+\frac{1}{2}\Delta] & \Delta & [(1+\frac{1}{2}\Delta) K] & \Delta K & [(1+\frac{1}{2}\Delta) K^2] & \Delta K^2 \ldots
\end{array}$$

Here 1 corresponds to the original entry y_n in the table. The natural differences as directly formed are expressed simply, while those which are means of the entries immediately above and below are enclosed by [ ]. But while the symbols occurring in the columns to the right of the central column (representing the function itself) will be readily understood, the construction of the columns to the left must now be explained. The numbers in the first column to the left are such that their differences appear in the central column. Thus

$$(\Delta K^{-1} - \delta K^{-1})\, y_n = y_n, \quad \Delta K^{-1} y_n = y_n + \delta K^{-1} y_n$$

and when one number in this column is fixed, the rest are formed by adding successively (when proceeding downwards) the tabulated values of the function. The entries in this column therefore contain an additive arbitrary constant. The second column to the left is related to this first column in exactly the same way as the first column to the central column, and therefore contains another arbitrary constant, but is otherwise definite.

The use of four different operators in the table may seem excessive, since they are all expressible in terms of one. In fact

$$\Delta = e^{wD} - 1, \quad \delta = 1 - e^{-wD}, \quad k = \sinh wD, \quad K = 4\sinh^2 \tfrac{1}{2} wD$$

and this suggests another mode of development which has here been deliberately avoided. But all these operators have simple special meanings

and it is important to notice that $k\delta^{-1}$ and $(1+\frac{1}{2}\Delta)$ are equivalent, but quite distinct from Δk^{-1}, though in the complete table, in which the mean differences are filled in, they all three denote one vertical step downwards.

292. As with Δ^{-1} and the other operators, D^{-1} is such that $DD^{-1}=1$, or D, D^{-1} represent inverse operations. And since D represents differentiation, D^{-1} represents integration. Thus take the formula (24). The column Δk^{-1} being formed with an arbitrary constant, the right-hand side of the equation, operating on y_n, will produce a function (represented in tabular form) which is $w^{-1}D^{-1}(1+\Delta)^{\frac{1}{2}}y_n = w^{-1}D^{-1}y_{n+\frac{1}{2}}$. On the application of D or differentiation, this becomes $w^{-1}y_{n+\frac{1}{2}}$. Hence the meaning of the formula is

$$w^{-1}\int^{a+mw} y\,dx = (\Delta K^{-1} + \tfrac{1}{24}\Delta - \tfrac{17}{5760}\Delta K + \tfrac{367}{967680}\Delta K^2 - \ldots)\,y_n \quad \ldots(26)$$

where m is written for $n+\frac{1}{2}$. The lower limit is arbitrary. But the right-hand side also contains an arbitrary constant, and this constant can now be chosen so as to fix the lower limit of integration. For let this limit be $a+\frac{1}{2}w$. If then $m=\frac{1}{2}$, $n=0$ in (26)

$$0 = (\Delta K^{-1} + \tfrac{1}{24}\Delta - \tfrac{17}{5760}\Delta K + \tfrac{367}{967680}\Delta K^2 - \ldots)\,y_0 \quad \ldots\ldots(27)$$

and the value of $\Delta K^{-1}.y_0$ is now determined. With it the whole of the corresponding column can be definitely calculated by successive additions of the values of the function. When this is done, (26) represents the definite integral of y between the limits $a+\frac{1}{2}w$ and $a+(n+\frac{1}{2})w$.

Quite similarly the meaning of (22) is seen to be

$$w^{-1}\int^{a+nw} y\,dx = (kK^{-1} - \tfrac{1}{12}k + \tfrac{11}{720}kK - \tfrac{191}{60480}kK^2 + \ldots)\,y_n \quad \ldots(28)$$

where the lower limit is a when

$$0 = (kK^{-1} - \tfrac{1}{12}k + \tfrac{11}{720}kK - \tfrac{191}{60480}kK^2 + \ldots)\,y_0.$$

But the latter form is not convenient, because $kK^{-1}y_0$, which is hereby determined, is the mean of two numbers not yet known. Now

$$2kK^{-1}y_0 = \Delta K^{-1}y_0 + \delta K^{-1}y_0, \quad y_0 = \Delta K^{-1}y_0 - \delta K^{-1}y_0$$

and therefore

$$\Delta K^{-1}y_0 = (\tfrac{1}{2} + \tfrac{1}{12}k - \tfrac{11}{720}kK + \tfrac{191}{60480}kK^2 - \ldots)\,y_0 \quad \ldots\ldots\ldots(29)$$

Thus $\Delta K^{-1}.y_0$ is determined, and the calculation proceeds as in the previous case. It is to be noticed that, though (27) has been derived from (26) and (29) from (28), (26) can be used in conjunction with (29), giving a and $a+(n+\frac{1}{2})w$ as the limits of integration, or (28) with (27), giving $a+nw$ as the upper limit and $a+\frac{1}{2}w$ as the lower limit.

293. In a similar way (23) and (25) give the second integrals, thus

$$w^{-2}\int_b^{a+nw}\left[\int_c^x y\,dx\right]dx = (K^{-1} + \tfrac{1}{12} - \tfrac{1}{240}K + \tfrac{31}{60480}K^2 - \ldots)\,y_n \;\ldots\ldots(30)$$

$$w^{-2}\int_b^{a+mw}\left[\int_c^x y\,dx\right]dx = (1 + \tfrac{1}{2}\Delta)(K^{-1} - \tfrac{1}{24} + \tfrac{17}{1920}K - \tfrac{367}{193536}K^2 + \ldots)\,y_n \;\ldots\ldots\ldots\ldots(31)$$

where $m = n + \frac{1}{2}$ as before. The lower limit c of the subject of the second integration is arbitrary. But if the first summation column, on the left of the function y, has been based on (29), $c = a$; if it has been based on (27), $c = a + \frac{1}{2}w$. The lower limit b of the second integration is also arbitrary and corresponds with the additional arbitrary constant in the second summation column K^{-1}. The latter is easily determined by taking the case $b = a$, $n = 0$ of (30). Thus

$$0 = (K^{-1} + \tfrac{1}{12} - \tfrac{1}{240}K + \tfrac{31}{60480}K^2 - \ldots)\,y_0 \;\ldots\ldots\ldots\ldots\ldots(32)$$

This gives $K^{-1}y_0$, and the whole of the second summation column becomes determinate when the first column has been fixed. Or again, if the lower limit b is to be $a + \frac{1}{2}w$, (31) gives when $b = a + \frac{1}{2}w$, $m = \frac{1}{2}$, $n = 0$,

$$0 = (1 + \tfrac{1}{2}\Delta)(K^{-1} - \tfrac{1}{24} + \tfrac{17}{1920}K - \tfrac{367}{193536}K^2 + \ldots)\,y_0$$

or

$$K^{-1}y_0 = -\tfrac{1}{2}\Delta K^{-1}y_0 + (1 + \tfrac{1}{2}\Delta)(\tfrac{1}{24} - \tfrac{17}{1920}K + \tfrac{367}{193536}K^2 - \ldots)\,y_0 \;\ldots(33)$$

This is quite general whatever the value of c, or of $\Delta K^{-1}y_0$, may be. But as $c = b$ usually, (27) can be used in this case, and then

$$K^{-1}y_0 = \{\tfrac{1}{24}(1 + \Delta) - \tfrac{17}{5760}(3 + 2\Delta)K + \tfrac{367}{967680}(5 + 3\Delta)K^2 - \ldots\}\,y_0 \ldots(34)$$

When the second summation column is based on (34) and the first on (27) $x = a + \frac{1}{2}w$ is the common lower limit for the double integration. When (29) and (32) are used in forming these columns, $x = a$ is the common lower limit. In either case (30) and (31) give the values of the double integrals to the upper limits $x = a + nw$ and $x = a + (n + \frac{1}{2})\,w$ respectively.

No attention has been given here to the limitations of the method which are imposed by the conditions of convergence of the expansions employed. In general the question is settled in practice by obvious considerations. But for a critical estimate of the accuracy attainable it is clearly important.

294. There is also a trigonometrical form of interpolation, otherwise known as harmonic analysis, which is of great importance. This is intimately related to Fourier's series, and indeed amounts to the calculation of the coefficients of this expansion. It will be well to recall the principal properties of the series, which may be stated thus:

The sum of the infinite series

$$a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx)$$

(n a positive integer), where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

is $f(x)$ throughout the interval $0 < x < 2\pi$, provided $f(x)$ is continuous.

At any point x in the interval where $f(x)$ is discontinuous, the sum of the series is $\frac{1}{2}\{f(x-0)+f(x+0)\}$.

It is assumed that the number of finite discontinuities and the number of maxima and minima of $f(x)$ are finite. These conditions are more than sufficient and are always satisfied by the empirical functions of practical computation.

The expansion is unique in the sense that no other coefficients can make the given series represent the same function over the stated interval so long as n remains integral.

If the series is absolutely convergent for all real values of x it is also uniformly convergent. Its sum has then no discontinuities and has the same value at $x=0$ and $x=2\pi$.

The sum of the series is a periodic function, with the period 2π. If $f(x)$ is also periodic with the same period, it coincides with the sum of the series for all values of x, but otherwise the functions coincide only in the interval $0 < x < 2\pi$. If $f(x)=f(-x)=f(x+2\pi)$, $f(x)$ is represented by a Fourier series containing cosine terms only ($b_n=0$). If $f(x)=-f(-x)=f(x+2\pi)$, $f(x)$ is represented completely by a series containing sine terms only ($a_0=a_n=0$). Similarly an arbitrary function can be represented within the interval 0 to π either by a sine series or by a cosine series when one of the functions $\pm f(2\pi - x)$ is assigned to the interval π to 2π.

295. When the function is given—and the term function has here an exceptionally wide meaning—the coefficients in its expression as a Fourier's series can be calculated by a special kind of integrator, known as an Harmonic Analyser, of which several forms have been invented. But here the equivalent arithmetical processes will be considered.

When the function is represented by a definite number of distinct values it is obvious that only a finite number of terms in the series can be determined, and it is necessary to assume that the practical convergency of the series is such that the remainder after a certain point is negligible. Let the finite series be

$$u = a_0 + \sum_{i=1}^{n} (a_i \cos i\theta + b_i \sin i\theta)$$

with $2n+1$ corresponding pairs of values, $u=u_r$, $\theta=\theta_r$. From the linear equations

$$u_r = a_0 + \Sigma\,(a_i \cos i\theta_r + b_i \sin i\theta_r)$$

the coefficients a_0, a_i, b_i can be found in the ordinary way. It is also easy to represent the result by a formula analogous to Lagrange's formula of interpolation (15). But when $\theta_r = 2r\pi/(2n+1)$ the solution can be effected in a very simple way.

It is necessary to consider the sums of two very simple series. In the first place

$$\begin{aligned}\sum_{r=0}^{s-1} \sin r\alpha &= \sum_{0}^{s-1} \{\cos(r-\tfrac{1}{2})\alpha - \cos(r+\tfrac{1}{2})\alpha\}/2\sin\tfrac{1}{2}\alpha \\ &= \{\cos\tfrac{1}{2}\alpha - \cos(s-\tfrac{1}{2})\alpha\}/2\sin\tfrac{1}{2}\alpha \\ &= \sin\tfrac{1}{2}s\alpha\sin\tfrac{1}{2}(s-1)\alpha/\sin\tfrac{1}{2}\alpha\end{aligned}$$

and this is 0 if $\alpha = 2p\pi/s$. Even when $p = p's$, p and p' being both integers, and therefore $\sin\frac{1}{2}\alpha = 0$, this remains true, for every term of the series is then zero. Similarly

$$\begin{aligned}\sum_{r=0}^{s-1} \cos r\alpha &= \sum_{0}^{s-1} \{\sin(r+\tfrac{1}{2})\alpha - \sin(r-\tfrac{1}{2})\alpha\}/2\sin\tfrac{1}{2}\alpha \\ &= \{\sin(s-\tfrac{1}{2})\alpha + \sin\tfrac{1}{2}\alpha\}/2\sin\tfrac{1}{2}\alpha \\ &= \sin\tfrac{1}{2}s\alpha\cos\tfrac{1}{2}(s-1)\alpha/\sin\tfrac{1}{2}\alpha\end{aligned}$$

and this is 0 also if $\alpha = 2p\pi/s$, unless $p = p's$. In the latter case each term of the series is 1 and the sum is s. Thus both the series vanish for $\alpha = 2p\pi/s$, except the cosine series when $\alpha = 2p'\pi$.

296. Let $u = u_r$ be the value of the function corresponding to the value of the argument $\theta = r\alpha$. The series will not now be limited to a finite number of terms. Then

$$\begin{aligned}\sum_{r=0}^{s-1} u_r \cos jr\alpha &= a_0\sum_r \cos jr\alpha + \sum_i\sum_r (a_i\cos jr\alpha\cos ir\alpha + b_i\cos jr\alpha\sin ir\alpha) \\ &= a_0\sum_r \cos jr\alpha + \tfrac{1}{2}\sum_i\sum_r a_i\{\cos(i+j)r\alpha + \cos(i-j)r\alpha\}\end{aligned}$$

$$\begin{aligned}\sum_{r=0}^{s-1} u_r \sin jr\alpha &= a_0\sum_r \sin jr\alpha + \sum_i\sum_r (a_i\sin jr\alpha\cos ir\alpha + b_i\sin jr\alpha\sin ir\alpha) \\ &= \tfrac{1}{2}\sum_i\sum_r b_i\{\cos(i-j)r\alpha - \cos(i+j)r\alpha\}\end{aligned}$$

when $\alpha = 2\pi/s$, for all the sine terms vanish immediately in the sum with respect to r. The cosine terms also vanish in the sum unless j, $i+j$ or $i-j$ is a multiple of s (including zero). Thus, j having in succession all values from 1 to $\frac{1}{2}(s-1)$, or $\frac{1}{2}s$,

$$\left.\begin{aligned}\frac{1}{s}\sum_{r=0}^{s-1} u_r &= a_0 + \sum_{m=1} a_{ms}, \quad (j=0) \\ \frac{2}{s}\sum_{r=0}^{s-1} u_r\cos\frac{2jr\pi}{s} &= a_j + \sum_{m=1}(a_{ms-j} + a_{ms+j}) \\ \frac{2}{s}\sum_{r=0}^{s-1} u_r\sin\frac{2jr\pi}{s} &= b_j + \sum_{m=1}(b_{ms+j} - b_{ms-j})\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(35)$$

When s equidistant values, $u_0, \ldots, u_{s-1}$, $(u_s = u_0)$, are known the operations indicated on the left are easily performed. Then, if the series converges so rapidly that the higher coefficients can be neglected, $a_0, a_1, b_1, \ldots$ are determined, as far as $a_{\frac{1}{2}(s-1)}$, $b_{\frac{1}{2}(s-1)}$ if s is odd, and as far as $a_{\frac{1}{2}s}$, $b_{\frac{1}{2}s-1}$ if s is even. The lower coefficients will naturally be calculated much more accurately than the higher, for there is little reason to suppose $a_{\frac{1}{2}s+1}$ small in comparison with $a_{\frac{1}{2}s-1}$. But it is well to compute the higher coefficients as a practical test of convergence.

297. It is usually convenient to make s an even number, and indeed a multiple of 4, so as to divide the quadrants symmetrically. Let $s = 2n$ and let the terms of higher order than a_n, b_{n-1} be neglected. Then (35) become

$$a_0 = \frac{1}{2n}\sum_{r=0}^{2n-1} u_r, \quad a_j = \frac{1}{n}\Sigma u_r \cos\frac{jr\pi}{n}, \quad b_j = \frac{1}{n}\Sigma u_r \sin\frac{jr\pi}{n} \quad \ldots\ldots(36)$$

$(j = 1, 2, \ldots, n-1)$. When $j = n$,

$$\frac{1}{n}\Sigma(-1)^r u_r = 2a_n, \quad 0 = b_n - b_n$$

so that a_n is determined, but not b_n; and this is natural, for $2n$ coefficients in addition to a_0 cannot be derived from $2n$ values u_r.

Let $n-j$ be written for j in (36). Then

$$a_{n-j} = \frac{1}{n}\sum_{r=0}^{n-1} u_r \cos\left(r\pi - \frac{jr\pi}{n}\right) = \frac{1}{n}\Sigma(-1)^r u_r \cos\frac{jr\pi}{n}$$

$$b_{n-j} = \frac{1}{n}\Sigma u_r \sin\left(r\pi - \frac{jr\pi}{n}\right) = -\frac{1}{n}\Sigma(-1)^r u_r \sin\frac{jr\pi}{n}.$$

Hence

$$\tfrac{1}{2}(a_j + a_{n-j}) = \frac{1}{n}\left\{u_0 + u_2\cos\frac{2j\pi}{n} + \ldots + u_{2n-2}\cos\frac{2j(n-1)\pi}{n}\right\}$$

$$= \frac{1}{n}\left\{u_0 + (u_2 + u_{2n-2})\cos\frac{2j\pi}{n} + (u_4 + u_{2n-4})\cos\frac{4j\pi}{n} + \ldots\right\}$$

$$\tfrac{1}{2}(a_j - a_{n-j}) = \frac{1}{n}\left\{u_1\cos\frac{j\pi}{n} + u_3\cos\frac{3j\pi}{n} + \ldots + u_{2n-1}\cos\frac{(2n-1)j\pi}{n}\right\}$$

$$= \frac{1}{n}\left\{(u_1 + u_{2n-1})\cos\frac{j\pi}{n} + (u_3 + u_{2n-3})\cos\frac{3j\pi}{n} + \ldots\right\}$$

$$\tfrac{1}{2}(b_j + b_{n-j}) = \frac{1}{n}\left\{u_1\sin\frac{j\pi}{n} + u_3\sin\frac{3j\pi}{n} + \ldots + u_{2n-1}\sin\frac{(2n-1)j\pi}{n}\right\}$$

$$= \frac{1}{n}\left\{(u_1 - u_{2n-1})\sin\frac{j\pi}{n} + (u_3 - u_{2n-3})\sin\frac{3j\pi}{n} + \ldots\right\}$$

$$\tfrac{1}{2}(b_j - b_{n-j}) = \frac{1}{n}\left\{u_2\sin\frac{2j\pi}{n} + u_4\sin\frac{4j\pi}{n} + \ldots + u_{2n-2}\sin\frac{2j(n-1)\pi}{n}\right\}$$

$$= \frac{1}{n}\left\{(u_2 - u_{2n-2})\sin\frac{2j\pi}{n} + (u_4 - u_{2n-4})\sin\frac{4j\pi}{n} + \ldots\right\}$$

($j=1, 2, \ldots, n-1$); and

$$a_0 + a_n = \frac{1}{n}(u_0 + u_2 + u_4 + \ldots + u_{2n-2})$$

$$a_0 - a_n = \frac{1}{n}(u_1 + u_3 + u_5 + \ldots + u_{2n-1}).$$

By this arrangement a_{n-j}, b_{n-j} are calculated together with a_j, b_j with scarcely more trouble than a_j, b_j alone. As a practical check on the convergence of the series these higher harmonics should be found.

298. The arrangement can be greatly simplified in special cases. For example, in the case $s=12$, $n=6$, let the data be arranged thus:

	u_0	u_1	u_2	u_3	u_4	u_5	u_6
		u_{11}	u_{10}	u_9	u_8	u_7	
Sums:	v_0	v_1	v_2	v_3	v_4	v_5	v_6
Differences:		w_1	w_2	w_3	w_4	w_5	

	v_0	v_1	v_2	v_3	w_1	w_2	w_3
	v_6	v_5	v_4		w_5	w_4	
Sums:	p_0	p_1	p_2	p_3	r_1	r_2	r_3
Differences:	q_0	q_1	q_2		s_1	s_2	

The equations for the coefficients are

$$\tfrac{1}{2}(a_j + a_{6-j}) = \tfrac{1}{6}(v_0 + v_2 \cos \tfrac{1}{3}j\pi + v_4 \cos \tfrac{2}{3}j\pi + v_6 \cos j\pi)$$
$$\tfrac{1}{2}(a_j - a_{6-j}) = \tfrac{1}{6}(v_1 \cos \tfrac{1}{6}j\pi + v_3 \cos \tfrac{1}{2}j\pi + v_5 \cos \tfrac{5}{6}j\pi)$$
$$\tfrac{1}{2}(b_j + b_{6-j}) = \tfrac{1}{6}(w_1 \sin \tfrac{1}{6}j\pi + w_3 \sin \tfrac{1}{2}j\pi + w_5 \sin \tfrac{5}{6}j\pi)$$
$$\tfrac{1}{2}(b_j - b_{6-j}) = \tfrac{1}{6}(w_2 \sin \tfrac{1}{3}j\pi + w_4 \sin \tfrac{2}{3}j\pi).$$

Hence two cases, according as j is even or odd:

	j even	j odd
$\tfrac{1}{2}(a_j + a_{6-j}) =$	$\tfrac{1}{6}(p_0 + p_2 \cos \tfrac{1}{3}j\pi)$	$\tfrac{1}{6}(q_0 + q_2 \cos \tfrac{1}{3}j\pi)$
$\tfrac{1}{2}(a_j - a_{6-j}) =$	$\tfrac{1}{6}(p_1 \cos \tfrac{1}{6}j\pi + p_3 \cos \tfrac{1}{2}j\pi)$	$\tfrac{1}{6} q_1 \cos \tfrac{1}{6}j\pi$
$\tfrac{1}{2}(b_j + b_{6-j}) =$	$\tfrac{1}{6} s_1 \sin \tfrac{1}{6}j\pi$	$\tfrac{1}{6}(r_1 \sin \tfrac{1}{6}j\pi + r_3 \sin \tfrac{1}{2}j\pi)$
$\tfrac{1}{2}(b_j - b_{6-j}) =$	$\tfrac{1}{6} s_2 \sin \tfrac{1}{3}j\pi$	$\tfrac{1}{6} r_2 \sin \tfrac{1}{3}j\pi$

and these forms can easily be made more general.

Then, for $j=2$,

$$\tfrac{1}{2}(a_2+a_4)=\tfrac{1}{6}(p_0-\tfrac{1}{2}p_2), \qquad \tfrac{1}{2}(b_2+b_4)=\tfrac{1}{6}s_1\cos 30^\circ$$

$$\tfrac{1}{2}(a_2-a_4)=\tfrac{1}{6}(\tfrac{1}{2}p_1-p_3), \qquad \tfrac{1}{2}(b_2-b_4)=\tfrac{1}{6}s_2\cos 30^\circ$$

for $j=1$,

$$\tfrac{1}{2}(a_1+a_5)=\tfrac{1}{6}(q_0+\tfrac{1}{2}q_2), \qquad \tfrac{1}{2}(b_1+b_5)=\tfrac{1}{6}(\tfrac{1}{2}r_1+r_3)$$

$$\tfrac{1}{2}(a_1-a_5)=\tfrac{1}{6}q_1\cos 30^\circ, \qquad \tfrac{1}{2}(b_1-b_5)=\tfrac{1}{6}r_2\cos 30^\circ$$

for $j=3$,

$$a_3=\tfrac{1}{6}(q_0-q_2), \qquad b_3=\tfrac{1}{6}(r_1-r_3)$$

and finally, for $j=0$,

$$a_0+a_6=\tfrac{1}{6}(p_0+p_2), \qquad a_0-a_6=\tfrac{1}{6}(p_1+p_3).$$

The calculation of the required terms is therefore extremely simple. The case when $s=24$, $n=12$, is almost equally so, but would require more space to exhibit in detail.

299. The mode of solution for the harmonic coefficients can be considered from another point of view. Let the s equidistant values $u_0, u_1, \ldots, u_{s-1}$ be given as before, and let the first p harmonics—including a_p, b_p—be required. If $2p=s-1$, the number of unknowns is equal to the number of values and the solution is unique. If $2p<s-1$, the number of equations is in excess of the number of coefficients to be determined. The latter can then be found by the rule of least squares, that is, so as to make the sum of the squared residuals a minimum. The equations being of the form

$$u_r=a_0+\sum_{i=1}^{p}\left(a_i\cos\frac{2ir\pi}{s}+b_i\sin\frac{2ir\pi}{s}\right)$$

the quantity which is to be made a minimum is

$$U=\sum_{r=0}^{s-1}\left\{a_0+\sum_{i=1}^{p}\left(a_i\cos\frac{2ir\pi}{s}+b_i\sin\frac{2ir\pi}{s}\right)-u_r\right\}^2.$$

The conditions are

$$\frac{\partial U}{\partial a_0}=\frac{\partial U}{\partial a_j}=\frac{\partial U}{\partial b_j}=0, \qquad (j=1,\ldots,p)$$

which, being $2p+1$ in number, determine a_0 and the $2p$ coefficients. They give in fact

$$\sum_{r=0}^{s-1}\left\{a_0+\sum_{i=1}^{p}\left(a_i\cos\frac{2ir\pi}{s}+b_i\sin\frac{2ir\pi}{s}\right)-u_r\right\}=0$$

$$\sum_{r=0}^{s-1}\cos\frac{2jr\pi}{s}\left\{a_0+\sum_{i=1}^{p}\left(a_i\cos\frac{2ir\pi}{s}+b_i\sin\frac{2ir\pi}{s}\right)-u_r\right\}=0$$

$$\sum_{r=0}^{s-1}\sin\frac{2jr\pi}{s}\left\{a_0+\sum_{i=1}^{p}\left(a_i\cos\frac{2ir\pi}{s}+b_i\sin\frac{2ir\pi}{s}\right)-u_r\right\}=0.$$

But since $2p < s-1$, $0 < j < p+1$ and $0 < i < p+1$, neither i nor $i+j$ is a multiple of s (including 0). Hence the only terms which do not vanish in the sum with respect to r arise when $i-j=0$, and therefore the equations become

$$sa_0 - \sum_{r=0}^{s-1} u_r = 0$$

$$\tfrac{1}{2}sa_j - \sum_{r=0}^{s-1} u_r \cos\frac{2jr\pi}{s} = 0, \quad \tfrac{1}{2}sb_j - \sum_{r=0}^{s-1} u_r \sin\frac{2jr\pi}{s} = 0$$

($j = 1, \ldots, p$). But these are identical with the earlier equations of the group (35) when the distant harmonics are omitted. Hence the harmonics to any order p derived by the general rule (36) from $2n$ equidistant values ($p < n$) are the same as would result from a least-square solution. Thus if the function is represented by a curve and the coefficients are calculated by the rule, a_0 gives the best horizontal straight line, $a_0 + a_1 \cos\theta + b_1 \sin\theta$ the closest simple sine curve, and so on, in the sense defined. This important property emphasises the independence with which the several coefficients are determined. Each apart from the rest is found with the greatest possible accuracy from the data according to the principle of least squares.

300. The method can be extended to the development of a periodic function in two variables,

$$F = \Sigma a_{ij} \sin(i\theta + j\theta' + \alpha).$$

For this may be written

$$F = a_0 + \sum_i (a_i \cos i\theta + b_i \sin i\theta)$$

where a_0, a_i, b_i are each of the same form as F with θ' in the place of θ. With any particular value of θ' and $2n$ equidistant values of F in respect to θ, a_0, a_i, b_i can be determined according to the rule expressed by (36). Each of these is a function of the chosen value of θ', and if the process is repeated with $2n$ equidistant values of θ', each coefficient can be expressed in the form

$$a_j = \alpha_0 + \sum_i (\alpha_i \cos i\theta' + \beta_i \sin i\theta')$$

by the same rule. When these expressions are inserted in the second form of F, the first form is readily deduced. This method was employed by Le Verrier in his theory of Saturn.

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